SCALING LAWS OF SAMPLED DATA AND MANY-TO-ONE CAPACITY OF SENSOR NETWORKS ON CLOSED SURFACES

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ABSTRACT

In this paper, we study the feasibility of sensing a bandlimited random field on a closed surface using a dense sensor network of inexpensive sensor nodes. The goal is to periodically gather quantized samples from the field and communicate it to a central collector, where this data is used to reconstruct the field. We show that under an arbitrary irregular placement of sensors that is uniformly discrete, the field can be reconstructed with a pointwise distortion of $O(\frac{1}{N})$ *when the sensors are equipped with* $\log N$*-bit quantizers.*

This irregular sampling result allows us, using a recent result in oversampled A/D conversion, to trade-off one $\log N$*-bit sensor to* N *one-bit sensors, and still obtain a pointwise distortion of* $O(\frac{1}{N})$*. The total data generated in the network also remains* $\Theta(\log N)$*. We then study the capacity of the many-to-one wireless network through which the sensor nodes communicate their data to a central collector. We find that, upto a constant, the capacity has the same scaling law in the number of nodes as the total data rate of the network. Hence such a network for both sensing and communicating is indeed feasible.*

Keywords: *Data gathering sensor networks, sampling and reconstruction.*

1. Introduction

In this paper, we study the feasibility of sensing a random field using a dense sensor network of inexpensive sensor nodes. The scenario we model and analyze is one in which a large number of sensors are deployed for data collection. The processing of data is performed at a central location, and the individual sensors are required to communicate their data to this location. It is natural to assume the lack of any wireless infrastructure in such networks. Also, it is desirable to make the sensor nodes as simple as possible. We therefore assume that the communication is on a shared wireless channel.

There are two aspects of the sensing problem in the situation outlined above: data acquisition and communication. For the first aspect, it is important to characterize the rate at which data must be sampled for some distortion metric to be met. The fact that data in a real sensing field becomes increasingly correlated as the distance between sensing locations decreases offers an opportunity for the network designer to reduce the total data generated by the network using distributed source-coding techniques. The second aspect is important because the nodes in a sensor networks are also assigned the function of communicating the data to the central collector. These two aspects are therefore not independent, and one natural question is: is the communication network formed by the sensing nodes capable of transferring the data generated to the collector node? This is all the more pertinent in a many source, single data-sink situation such as the one described here: there is a communication bottleneck at the data collector. Indeed, if one assumes a model where all nodes, including the sink, can receive only a constant number of bits per unit time, then the data rate assigned to any sensing node is inversely proportional to the number of nodes. Depending on the distortion metric, this might not be sufficient to support the data rate required for the desired level of performance [1].

Several recent papers have studied problems lying in the broad area outlined above, with varied models and conclusions. The authors in [1] take a rate-distortion approach to quantify the data required to attain a given mean square distortion, and prove that if the central node can receive only a constant number of bits per unit time, then the many-to-one capacity is insufficient to support the required data rate. The same data generation model is assumed in [2], however, using information theoretic channel capacity results, the authors prove that the channel capacity is adequate for the communication needs of the network. A similar sensing problem is studied in [3], where the authors consider the reconstruction of a bandlimited wide sense stationary (WSS) random field using quantized samples. The analysis is based on the recent fundamental signal processing results developed in [4]. Again an negative conclusion on the feasibility is indicated.

We take an approach similar to the one in [3] in this paper, and study both the data acquisition and communication aspects of the problem in detail. However, in the specific sensing scenario that we study, we answer the feasibility question posed above *in the affirmative*.

Specifically, we consider a sensing model in which sensors are deployed on a closed curve (such as a circle), or certain closed surfaces (such as a torus). The field to be

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sensed is a stationary bandlimited random field defined on the closed curve. Such sensor networks arise, for example, when sensing any closed boundary, such as the coastline of an island, or inside a particle accelerator.

For simplicity, we assume that the location of the collector is at a point inside the curve, and the maximum distance of the data collector from any sensor node is bounded. We prove that stable reconstruction of the field can be performed using samples taken at a fixed number of arbitrary non-uniform locations on the circle. The stability here is in the sense that small perturbations in the sample values (such as those caused by quantization error) lead to a pointwise error of at most the same order. We then use the stable reconstruction schemes to prove that we can exchange a small number of high precision sensors with a large number of low precision sensors, with the total data rate being unchanged. These results also extend those of [3] and [4] to the periodic random field setting, when the field is no longer square integrable.

On the communication side, we model the channel as a shared additive white Gaussian noise (AWGN) channel, as in [2], [5]. This assumes that each node is able to hear an attenuated version of the transmission of every other node. We discuss the capacity of the many-to-one channel under an individual power constraint on each of the sensors' transmissions, and show that it scales as the logarithm of the number of transmitters. Thus, the capacity has the same scaling law as the data generated by the network, which proves the feasibility of the sensing operation, given that the constants in the scaling (both the capacity and the data generated scale as $\log N$, where N is the number of nodes in the network) are adjusted appropriately.

The outline of this paper is as follows. We study the data acquisition problem in Section 2 and Section 3. Section 2 considers the case with a fixed number of sensors equipped with high precision quantizers while Section 3 is focused on the case when sensors are equipped with one-bit quantizers. The communication problem is then studied in Section 4. We make some concluding remarks in Section 5.

2. Sampling and reconstruction

In this section we quantify the data that is to be communicated to the central data-processing unit for reconstruction of the field to be sensed. We start with specifying the the field models and the distortion criterion. In this section, we find the data rate for the case when we use sensors equipped with accurate ($\log N$-bit, for some integer N) quantizers. Later, in Section 3, we prove that the data rate (and the reconstruction error, upto a proportionality constant) remains the same when we replace one $\log N$-bit quantizer with N one-bit quantizers.

2.1. Model of the field to be sensed

We would present results on the sampling and reconstruction of both deterministic functions and random processes, defined on closed curves. The feature of a function or a random process defined on a closed curve that is relevant to our study is that it is forced to be periodic. Since the maximum distance from the data collector to any sensor is a bounded, we can consider, without lose of generality, the closed curve to be a circle, and the location of the center to be center of the circle.

We follow notiation similar to that in [3]. Denote a deterministic field by a function $f(t)$ and a random field by the process $X(t)$, where t denotes the position on the closed curve to be sensed. The length of the perimeter of the circle is assumed to be L. When we "open up the circle", we can extend the field to a periodic field on the real line, with the period being L. Denote the "opened up" circle on the line by the interval $[-\frac{L}{2}, \frac{L}{2}]$. We then have for the extended field, $f(t) = f(t \mod L)$ and $X(t) = X(t \mod L)$.

The distortion that we seek to minimize is the pointwise error in reconstruction for deterministic functions,

$$\sup_{t\in[-\frac{L}{2},\frac{L}{2}]} |f(t) - f^Q(t)|,$$

and the pointwise mean square error for random fields,

$$\sup_{t\in[-\frac{L}{2},\frac{L}{2}]} E\left[\left|X(t) - X^Q(t)\right|^2\right].$$

where $f^Q(t)$ and $X^Q(t)$ denote respectively the corresponding reconstructions. We further study the data rate required for any given value of the distortion.

Deterministic case: Let $\omega_o = \frac{2\pi}{L}$. We assume that the function f is bandlimited to $[-B\omega_o, B\omega_o]$, where B is an integer. Also, we assume that f is continuous, and has dynamic range $[-1, 1]$, that is $|f| \leq 1$. As observed above, we can think of f as being defined on the real line as a periodic function with period L. Note that the Fourier transform of f consists of impulses at intervals of ω_o.

Stochastic case: We assume that the random process $X(t)$ is wide sense stationary (WSS). Since X is periodic, the autocorrelation function $R_X(\tau) = E[X(t)X(t+\tau)]$ is also periodic with period L. We assume that the process is bandlimited to $[-B\omega_o, B\omega_o]$ as before. Further, we assume that X has dynamic range $[-A, A]$ for some $A < \infty$.

2.2. Uniform sampling

In this section, we illustrate the reconstruction of a deterministic function f from samples taken at regular intervals (throughout this paper, uniform sampling refers to sampling at regular spaced intervals). Further, we bound the error in reconstruction that results if these samples are quantized using a finite number of bits.

The Nyquist sampling theorem [6] requires that the sampling intervals be strictly shorter than $\frac{\pi}{B\omega_o} = \frac{L}{2B}$. Thus, we require at least $(2B+1)$ sampling points, and the length of the sampling interval is given by $\lambda = \frac{L}{2B+1}$. We would refer to λ as the Nyquist interval in what follows. Denote the locations of the sensors by $t_k = \dfrac{k}{\lambda}$, $k = -B, \ldots, B$.

The reconstruction formula for the function from the samples taken at the locations t_k follows from well known results in signal processing [6]. Specifically, if the Fourier series of the function is given by $f(t) = \sum_{k=-B}^{B} a_k e^{jk\omega_o t}$, we know that the function is completely described by the $2B+1$ coefficients a_k. Denote $\vec{\boldsymbol{a}} = (a_{-B}, ..., a_B)^T$ the vector of coefficients and $\vec{\boldsymbol{t}} = (t_{-B}, ..., t_B)^T$ the vector of samples. Further, let $f(\vec{\boldsymbol{t}}) = (f(t_{-B}), ..., f(t_B))^T$. Then, we have

$$f(\vec{\boldsymbol{t}}) = U \cdot \vec{\boldsymbol{a}}, \tag{1}$$

where

$$U = \begin{bmatrix} e^{-jB\omega_o t_{-B}} & e^{-j(B-1)\omega_o t_{-B}} & \dots & e^{jB\omega_o t_{-B}} \\ \vdots & \vdots & & \vdots \\ e^{-jB\omega_o t_B} & e^{-j(B-1)\omega_o t_B} & \dots & e^{jB\omega_o t_B} \end{bmatrix}.$$

It is easily checked that U is a DFT (Discrete Fourier Transform) matrix, which is known to be pseudo-unitary and has an inverse given by $\dfrac{1}{2B+1}U^*$ [6]. Inverting (1), we get

$$a_k = \frac{1}{2B+1} \sum_{i=-B}^{B} e^{-j\frac{2\pi}{2B+1}ik} f(t_i). \tag{2}$$

Suppose the sensors are equipped with $\log N$-bit quantizers. The fact that the samples taken by the quantizers are accurate only up to $\log N$ bit precision, leads to distortion in reconstruction. In what follows, we quantify this distortion.

Denote the quantized samples by $\hat{f}(t_k)$, and the values of the coefficients of the Fourier series expansion found by using using $\hat{f}(t_k)$ instead of $f(t_k)$ in (2) by $\hat{a}_k$. We then have distorted coefficients $\hat{a}_k$ based on these quantized samples, where $\hat{a}_k = \frac{1}{2B+1} \sum_{i=-B}^{B} e^{-j\frac{2\pi}{2B+1}ik} \hat{f}(t_i)$. The reconstructed function then becomes:

$$f^Q(t) = \frac{1}{2B+1} \sum_{k=-B}^{B} \sum_{i=-B}^{B} e^{-j\frac{2\pi}{2B+1}ik} \hat{f}(t_i) e^{jk\omega_o t}. \tag{3}$$

Since the sensors are equipped with $\log N$-bit quantizers, the worst case quantization error in at any sampling location t_i is $|X(t_i) - \hat{X}(t_i)| \leq \epsilon_Q = \frac{1}{N}$.

Since the locations of the sampling points are fixed, and since there are only a finite number of interpolation points, bounded error in the sample values then leads to a bounded error in reconstruction at any point. It is easily checked that $|f(t) - f^Q(t)| \leq (2B+1)\epsilon_Q$, which leads to the following result on the scaling laws of the data rate under uniform sampling.

Lemma 1 *Consider a deterministic function f defined on the circle $[-\frac{L}{2}, \frac{L}{2}]$. If the sensors are equipped with $\log N$-bit quantizers and regularly spaced, then*

$$|f(t) - f^Q(t)| \leq \frac{2B+1}{N}, \; \forall \, t \in [-\frac{L}{2}, \frac{L}{2}].$$

That is, the pointwise reconstruction error is $O(\frac{1}{N})$. The bit-rate required is $\log N$ per Nyquist interval, or $R_{unf} = \lambda k = \lambda \log N$ bits per unit length. In terms of the bit rate R_{unf}, the reconstruction error is $O(2^{-\frac{R_{unf}}{\lambda}})$.

We show in the following section that the same scaling laws can be obtained even if the locations of the samples are not uniformly spaced. To avoid repetition, we defer the reconstruction of WSS processes from their samples to Section 2.4, where we prove a general formula for reconstruction from arbitrarily spaced samples.

2.3. Non-uniform sampling

In this section, we generalize the results of Section 2.2 to irregularly placed sensors. Non-uniform sampling is useful as it is much simpler in practice to randomly scatter the sensors in the field. More importantly, with a reconstruction scheme robust to quantization errors in non-uniform samples, we can use it to trade off the accuracy of the samples to their density (or the oversampling rate [4]), which then enable us to design networks with simple(single-bit precision) and inexpensive sensors instead of a few sensor with accurate and expensive quantizers. The details of the oversampling scheme are presented in Section 3.

The theory of non-uniform sampling is well developed, and there are a number of results [7] on the density of samples that is required to ensure reconstruction. However, most of these results are in terms of real valued samples, and it was only recently that a stability result for reconstruction from non-uniform samples of square integrable (that is, finite energy) bandlimited signals was published [4]. We prove here a similar result for periodic (and hence, infinite energy) bandlimited functions. Our approach is fully independent of the one taken in [4], though some similarities can be observed in the expressions that we deal with.

2.3.1. Deterministic fields

Under non-uniform sampling, the required density of the sampling points is known to be the same as for uniform sampling [7], [8], and therefore the required *number* of samples remains the same. So, consider again $(2B+1)$ samples of the function at locations $\{q_k, k = -B, ..., B\}$ in the interval $[-\frac{L}{2} \frac{L}{2}]$. Further, assume that

$$\min_{m,l,m \neq l} |(q_m - q_l) \mod L| = d_{min} > 0, \tag{4}$$

that is, the points $\{q_k\}$ are *uniformly discrete*.

Denote $\vec{\boldsymbol{q}} = (q_{-B}, ..., q_B)^T$ the vector of samples, and let $f(\vec{\boldsymbol{q}}) = (f(q_{-B}), ..., f(q_B))^T$. Then from the Fourier series expansion we get,

$$f(\vec{\boldsymbol{q}}) = V \cdot \vec{\boldsymbol{a}}, \tag{5}$$

where

$$V = \begin{bmatrix} e^{-jB\omega_o q_{-B}} & e^{-j(B-1)\omega_o q_{-B}} & \dots & e^{jB\omega_o q_{-B}} \\ \vdots & \vdots & & \vdots \\ e^{-jB\omega_o q_B} & e^{-j(B-1)\omega_o q_B} & \dots & e^{jB\omega_o q_B} \end{bmatrix}.$$

We can factorize V as $V = RW$, where R is a diagonal matrix,

$$R = \mathrm{diag}(e^{-jB\omega_o q_{-B}}, e^{-jB\omega_o q_{-(B-1)}}, \dots, e^{-jB\omega_o q_B}),$$

and W is a Van der Monde matrix,

$$W = \begin{bmatrix} 1 & y_{-B} & \cdots & y_{-B}^{2B} \\ 1 & y_{-(B-1)} & \cdots & y_{-(B-1)}^{2B} \\ \vdots & \vdots & & \vdots \\ 1 & y_B & \cdots & y_B^{2B} \end{bmatrix}$$

with $y_k = e^{j\frac{2\pi}{L}q_k}$. It is well known that a Van der Monde matrix with distinct entries y_k is invertible [9]. Since R too is invertible, $V^{-1} = W^{-1}R^{-1}$ exists. Thus the coefficients a_k's can be obtained by inverting (5).

When the available samples are quantized, denoted by $\hat{f}(q_k)$, we can derive similarly (as in the previous subsection) the reconstruction function:

$$f^Q(t) = \sum_{k=-B}^{B}\sum_{l=-B}^{B}[W^{-1}]_{kl}e^{jk\omega_o t}e^{j\omega_o Bq_l}\hat{f}(q_l).$$

We then have the following bound on the reconstruction error and scaling laws of the data rate under non-uniform sampling.

Theorem 1 *Under non-uniform sampling such that (4) is satisfied, if the sensors are equipped with* $\log N$*-bit quantizers, the pointwise reconstruction error is bounded as*

$$|f(t) - f^Q(t)| \le \frac{C}{N},$$

where C *is a constant that depends only on* d_{min}*. The scaling of the distortion and the data rate is the same as in the case of uniform sampling as specified in Lemma 1.*

Proof. As before, the quantization error in any sample is bounded, i.e. $|X(q_l) - \hat{X}(q_l)| \le \epsilon_Q = \frac{1}{N}$ for all l. Then,

$$\begin{aligned} & |f(t) - f^Q(t)| \\ = & \left|\sum_{k=-B}^{B}\sum_{l=-B}^{B}[W^{-1}]_{kl}e^{jk\omega_o t}e^{j\omega_o Bq_l}\left(f(q_l) - \hat{f}(q_l)\right)\right| \\ \le & \epsilon_Q \sum_{k=-B}^{B}\sum_{l=-B}^{B}\left|[W^{-1}]_{kl}\right|. \end{aligned} \quad (6)$$

Thus, it suffices to bound the entries of W^{-1}. These entries can be expressed as

$$[W^{-1}]_{kl} = \frac{(-1)^k\sigma_{((2B+1)-k)}(y_{(-B)},\ldots,y_{l-1},y_{l+1},\ldots,y_B)}{y_l\prod_{m=-B,m\neq l}^{B}\left(\frac{y_m}{y_l}-1\right)},$$

where $\sigma_n(.)$ is the symmetric polynomial of degree n (in $2B$ variables), that is, a polynomial which is unchanged by any permutation of its variables. Each y_k is a complex exponential (and hence so are the products of the elements in $\{y_k\}$), and the number of terms in $\sigma_{((2B+1)-k)}(.)$ is given by $\begin{pmatrix} 2B \\ 2B+1-k \end{pmatrix}$ Therefore, using the triangle inequality, the numerator can be bounded in magnitude by

$$\begin{pmatrix} 2B \\ 2B+1-k \end{pmatrix} \le \begin{pmatrix} 2B \\ B \end{pmatrix}. \quad (7)$$

Also, each term in the product in the denominator can be bounded in magnitude as follows:

$$\begin{aligned} \left|\frac{y_m}{y_l} - 1\right| &= \left|\exp\left(j\frac{2\pi}{L}\{q_m - q_l\}\right) - 1\right| \\ &= 2\left|\sin\left(\frac{\pi}{L}\{q_m - q_l\}\right)\right| \ge 2\left|\sin\left(\frac{\pi}{L}d_{min}\right)\right| \end{aligned} \quad (8)$$

where d_{min} is as in (4).

From (7) and (8) we conclude that each term in W^{-1}, and hence the sum in (6), can be upper bounded by a constant that depends only on d_{min}. ∎

2.4. Random stationary fields

In this section we prove a reconstruction formula for a band-limited wide sense stationary periodic field, and bound the expected reconstruction error. These are the stochastic analogues of the results in Section 2.3.1.

We first give an interpolation formula using real-valued samples (thus no quantization error), and then derive the reconstruction error resulting from quantized samples. The proof has been omitted due to considerations of paper size.

Lemma 2 *Consider a band-limited periodic WSS process* X *as in Section 2.1. Let*

$$X^I(t) = \sum_{k=-B}^{B}\sum_{l=-B}^{B}[W^{-1}]_{kl}e^{j\omega_o Bq_l}X(q_l)e^{jk\omega_o t}. \quad (9)$$

Then, $E\left[\left|X(t) - X^I(t)\right|^2\right] = 0.$

When available samples are quantized, denoted by $\hat{f}(q_k)$, the interpolation given by (9) then has to be based on the quantized samples, and the reconstruction function becomes:

$$X^Q(t) = \sum_{k=-B}^{B}\sum_{l=-B}^{B}[W^{-1}]_{kl}e^{j\omega_o Bq_l}\hat{X}(q_l)e^{jk\omega_o t}.$$

We then have the following:

Theorem 2 *For a wide sense stationary periodic field, if the sensors are equipped with* $\log N$*-bit quantizers, the pointwise mean square reconstruction error is bounded as*

$$E\left[\left|X(t) - X^Q(t)\right|^2\right] \le \frac{C'}{N^2},$$

where C' *is a constant. That is, the pointwise reconstruction error is* $O(\frac{1}{N^2})$*. The data rate required is* $\lambda \log N$ *per unit length per snapshot of the field, or* $\frac{\log N}{N}$ *per sensor. In terms of the bit rate* R_{unf}*, the reconstruction error is* $O(2^{-\frac{R_{unf}}{\lambda}})$*.*

This theorem follows because the quantization error in any sample is, as before, bounded by $\epsilon_Q = \frac{1}{N}$, and because the entries of W^{-1} are bounded as we proved in the previous section.

3. Data gathering with densely placed one-bit resolution sensors

For sensing applications it is preferable and also more robust to have several inexpensive, simple(low-bit) sensing nodes rather than few expensive(high-bit) nodes. The results of irregular sampling obtained in previous sections can be applied for the purpose of designing such sensor networks with a large number of low-bit and inexpensive sensors instead of using nodes equipped with high precision A/D converters. In this section, we show that one can trade off the accuracy of the samples by their density, and still get reconstruction error of the same order.

The core idea is based on a recent result [4] in oversampled A/D conversion. Briefly, the approach is to add a sufficiently smooth dither function d to the function f that is to be sensed. Then, if one takes closely spaced one-bit samples of the function $f+d$, one can find the zero crossings of this function to high spatial accuracy. This in turn translates to high accuracy in the reconstruction of f.

This above idea has already been applied in [3] to sensor networks in square-integrable random fields. We take the same approach as in [4] here and apply to periodic (thus non-square-integrable) random fields such as closed surfaces. We show that using $(2B+1)N$ one-bit sensors leads to reconstruction error of the same order as using $(2B+1)$ sensors of $\log N$-bit precision.

3.1. Deterministic functions

Denote by I_k the interval $\left(t_k - \frac{\lambda}{2}, t_k + \frac{\lambda}{2}\right)$, with $t_k = \frac{k}{\lambda}$, and $k = -B, \ldots, B$. In each of these $(2B+1)$ intervals, we place N one-bit sensors at a uniform spacing of $\tau = \frac{1}{N\lambda}$.

Define a dither function $d(.)$ on the circle such that

1. $\left|d\left(\frac{k}{\lambda} + \frac{1}{2\lambda}\right)\right| \geq \gamma > 1$,
2. $\text{sgn}\left[d\left(\frac{k}{\lambda} - \frac{1}{2\lambda}\right)\right] = -\text{sgn}\left[d\left(\frac{k}{\lambda} + \frac{1}{2\lambda}\right)\right]$,
3. d is continuously differentiable in the intervals I_k, and $\max_k \sup_{t \in I_k} |d'(t)| \leq \Delta_d$,

for $k = -B, \ldots, B$. These conditions ensure that $f+d$ has at least one zero crossing in each interval I_k.

We now use the N one-bit sensors to detect the sign of $f+d$. Consider a specific interval I_k, number the N sensors as $1, \ldots, N$ from left to right. Each sensor can communicate with its nearest neighbour. Suppose m is the first sensor that detected a sign change. Then z_k, the actual zero crossing point of $f+d$ must lie in the interval $\left(t_k - \frac{1}{2\lambda} + m\tau, t_k - \frac{1}{2\lambda} + (m+1)\tau\right)$. Let q_k denote the midpoint of this interval. Since the derivative of f is bounded by $B\omega_o \sum_k |a_k| = \Delta_f$ (say), that of $f+d$ is bounded by $(\Delta_f + \Delta_d)$. So we have,

$$|(f+d)(q_k)| \leq (\Delta_f + \Delta_d)\frac{\tau}{2} = \left(\frac{\Delta_f + \Delta_d}{2}\right)\frac{1}{N} := \epsilon'_Q.$$

Hence, the maximum error in the samples $f(q_k)$ is ϵ'_Q. The samples $\{q_k\}$ form a uniformly discrete sequence since $\left|(f+d)\left(\frac{k}{\lambda} + \frac{1}{2\lambda}\right)\right| \geq \gamma - 1$ and $|(f+d)'|$ is bounded.

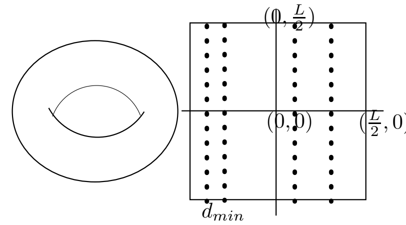

Fig. 1. "Opening up" a torus. The lines in the opened-up square indicate the densely sampled circles along the length of the torus.

Thus, the conditions of Section 2.3 are met, and we get a pointwise error of $O(\epsilon'_Q)$, that is $O(\frac{1}{N})$.

3.2. Stationary random processes

We can sample the sign changes of X using one bit sensors following the same procedure as outlined for deterministic functions in Section 3. However, as pointed out in [3], these need not necessarily correspond to zero-crossings of X (there might be discontinuities in the sample path of X and so there might not be a zero crossing of X between two points where the sign differs). However, since the set of points where the signs of X differ is a countable set, we can define a bandlimited process, X_{BL}, so that it agrees with X with probability one on this set, and X_{BL} also changes sign at the same locations. Since the derivative of X_{BL} is bounded uniformly, say by Δ_X, we get $|X_{BL}(q_l)+d(t_l)| \leq \frac{\Delta_X+\Delta_d}{2\lambda}\frac{1}{N}$.

Now, we can use the reconstruction

$$X^Q(t) = \sum_{k=-B}^{B}\sum_{l=-B}^{B}[W^{-1}]_{kl}(-d(q_l))e^{j\omega_o B q_l}e^{jk\omega_o t}.$$

and get $E\left|X(t) - X^Q(t)\right|^2 \leq C\frac{1}{N^2}$, based on Lemma 2.

In summary, we have proved that for a pointwise mean square error distortion of $O(\frac{1}{N^2})$, the data rate required is $\lambda \log N$ bits per unit length per snapshot of the field, or $\frac{1}{N}\log N$ bits per sensor. The distortion therefore decreases exponentially fast as the (total) data rate increases.

3.3. Extension to closed surfaces

The results obtained above for a single dimensional curve can be extended to two-dimensional closed surfaces that are "periodic" in both dimensions. For example, the natural extension of the circle to two dimensions is a torus. One possible application of this extension is to sensing inside particle accelerators. Again, we assume that the field to be sensed is band-limited and the data collector is located at the center of the torus (see Figure 1). Consider a torus that opens up into a square of side L, as shown in Figure 1. We can extend the definition of the function to the two-dimensional plane so that $f(r,s) = f(r \mod L, s \mod L)$. The ideas developed in the single-dimension case can be easily applied to the two-dimension case. Using one-bit sensors, we need to sample only one of the two dimensions densely. We can consider $2B+1$ discrete circles running "along the length" of the torus as illustrated in Figure 1. The only constraint on these circles is that the distance between any two of them (along the "width" of the torus) is bounded below by a constant. Each of these $2B+1$ circles can now be sampled densely using one-bit sensors, just as the one-dimensional circle considered above, and then function can then be reconstructed at any arbitrary point on the torus. If we sample each circle running along the length of the torus with N one bit sensors, then the total number of sensors used for sampling on the torus is $(2B+1)N$ and the total number of bits generated per unit area is again $\Theta(\log N)$.

4. Transport issues

We have proved that we can reconstruct the random process X defined on a closed curve using the first zero crossings of the sum $X+d$, where d is a known dither function. In this section, we focus on the problem of communicating the sampled data to the central collector. There are two parts of the communication problem: compression to remove the redundancy in the data, and communication over a shared, finite capacity channel to the data-processing unit.

4.1. Distributed data compression

We concentrate on the scheme in which we densely sample the field using one-bit sensors in this section. We disregard temporal correlation in the samples, that is the correlation across snapshots, and therefore come up with a conservative estimate of the data generated by the network. The temporal correlation could be used to further compress the data.

For each snapshot of the random field, each sensor detects the sign of the dithered field, setting its bit to 1 or 0. There might be more than one zero-crossings in some intervals, and we can remove this redundant data through local communication among the sensors. In each interval, each sensor communicates two bits to the node on its right, indicating both whether a zero crossing has been detected so far and the sign of its observation. By this two-bit rippling, we can set up the zero crossing data at the sensors so that only the node just before the first zero crossing has its bit set to 1, and all others have their bits set to 0. This local communication may be carried out in parallel and asynchronously with the communication with the central collector.

Since there are N possible locations of a zero-crossing in an interval I_k, the total number of bits needed to describe the sample in an interval is $\log N$. Using Slepian-Wolf coding [10], this rate can be achieved without collaboration between nodes. Recall from Section 2 and Section 3 that the distortion decays exponentially in the data rate.

We do not account for the local data communication in the communication capacity considerations below. This communication has a low demand on the communication resources, requiring only two bits to be rippled through the network using nearest neighbor communication. This simple local communication only causes marginal increase in data rate as the number of sensors becomes large.

4.2. Channel capacity

As observed before, a centralized data-gathering application such as the one we study suffers from a communication bottleneck at the data collector. However, we prove in this section that if the data collector is located at a bounded distance from all sensors, then the capacity of a shared wireless channel is large enough to support the required data rate.

4.2.1. Channel Model

Label the sensors with numbers from 1 to $N_T=(2B+1)N$ and the central collector with the number 0. We assume that the communication channels between the sensors and the collector node, and between the sensors themselves can be modeled as additive white Gaussian noise (AWGN) channels. Further, we take the so called far-field model for attenuation, so that the channel model is:

$$\boldsymbol{y}_k = \sum_{l=1, l\neq k}^{N_T} \frac{\boldsymbol{x}_l}{d_{kl}} + \boldsymbol{w}_k, \tag{10}$$

where $\boldsymbol{x}_l$ denotes the input of node l, $\boldsymbol{y}_k$ denotes the received symbol at node k, d_{kl} is the distance between nodes k and l. Further, we assume that *each* sensor is constrained in power, $E[|\boldsymbol{x}_l|^2] \leq P$. The variables $\boldsymbol{w}_k$ are assumed to be mutually independent, Gaussian with mean 0 and variance σ^2.

Finding the capacity of the many-to-one channel as presented above is a difficult and unsolved problem in network information theory. However, we are able to find upper and lower bounds for this capacity that scale identically.

4.2.2. Lower bound on the many to one channel capacity

To find a lower bound, we simply ignore the channels between the sensors. We are therefore left with only the channels between the sensors and the collector, thus ruling out any cooperation between the sensors. This is the well known Gaussian multiple-access channel (MAC), the capacity region of which is given by [11]:

$$\sum_{i\in S} R_i \leq \frac{1}{2}\log\left(1+\frac{\sum_{i\in S}\{P_i/(d_{i0}^2)\}}{\sigma^2}\right),$$

for each subset S of $\{1,\ldots,N_t\}$. Here R_i denotes the rate achieved by transmitter i. Any set of rates in the region defined by the above inequalities is achievable. Thus, the maximum achievable sum rate is given by

$$\begin{aligned}\sum_{i=1}^{N_T} R_i &= \frac{1}{2}\log\left(1+\frac{(N_T P)/d^2}{\sigma^2}\right)\\ &= \frac{1}{2}\log\left(1+\frac{(2B+1)NP/d^2}{\sigma^2}\right), \end{aligned} \tag{11}$$

where d is the radius of the circle. For a general closed curve (or surface), we can obtain a lower bound by taking d to be the distance of the sensor furthest away from the collector.

4.2.3. *Upper bound on the many to one capacity*

We prove in this section that the capacity can indeed not have a scaling better than the $\log N$ lower bound found in the previous section. To obtain an upper bound on the capacity, we assume perfect communication channels between the sensors, so that they can jointly decide their channel coding scheme. This is can be viewed as a multiple antenna channel with N_T input antennas (one at each sensor) and one receive antenna (at the collector). The capacity of this channel is also well known [12], and is given by

$$C = \frac{1}{2}\log\left(1+\frac{N_T^2 P}{\sigma^2}\right) = O(\log N), \tag{12}$$

using $N_T = (2B+1)N$. It is clear from (11) and (12) that the per-sensor capacity of the many-to-one channel is $\Theta(\frac{1}{N}\log N)$.

It is worth mentioning here that in [2], the same scaling law as above is proved with the *total* power assigned to the sensors being fixed. However, the derivation in [2] relies on "amplification" of power through reducing the distance between nodes. The received power, however, can be at most as much as the power that the transmitting node uses in its antenna. Also, the far-field model (see (10)) fails to model the channel reasonably as the distance between the nodes decreases. The result in [2] therefore appears to be true for moderately large values of N, and not asymptotically.

4.2.4. *Adjusting the constants in the scaling laws*

We have found both the capacity and the data generated by the network to have the same scaling law. The sensing operation would therefore be feasible for any non-zero distortion constraint, given that the multiplicative constants of the scaling laws (the constants multiplying $\log N$) can be adjusted so that the capacity is always larger than the sensing data to be sent. Two controls that are available to the network designer to ensure this are the bandwidth of the communication channel, and adding a polynomial (in N) number of relay nodes in the network.

5. Conclusion

We have proved that it is feasible to sense a bandlimited random field on a closed curve. Both the capacity and the data generated by the network scale as $\Theta(\log N)$, with N being the number of single-bit precision sensors deployed uniformly in each Nyquist interval on the curve. Our results can be extended to certain closed surfaces, and we have illustrated them through the example of a torus. Essentially, we require a surface that forces a periodic extension of the signal in both dimensions.

Our results extend the results of [4] and [3] for the sampling and reconstruction of periodic functions, and our analysis techniques are largely inspired by these papers.

While we present a positive result in this paper, there is still a gap between the many-to-one capacity and the data rate required for sensing in other geometries.

Other future research directions include: considering models of correlation of realistic random fields, both across points in space and instants of time, or finding schemes to utilize the resources of the channel for many-to-one communication efficiently.

6. References

[1] D. Marco, E. J. Duarte-Melo, M. Liu, and D. L. Neuhoff, "On the many-to-one transport capacity of a dense wireless sensor network and the compressibility of its data," in *Lecture notes in Comp. Sci.*, L. J. Guibas and F. Zhao, Eds., pp. 1–16. Springer, 2003.

[2] H. El Gamal, "On the scaling laws of dense wireless sensor networks," *submitted to IEEE Transactions on Information Theory*, April 2003.

[3] P. Ishwar, A. Kumar, and K. Ramchandran, "On distributed sampling in dense sensor networks: a "bit-conversation" principle," *IEEE Journal on Selected Areas in Communication*, July 2003.

[4] Z. Cvetkovic and I. Daubechies, "Single-bit oversampled A/D conversion with exponential accuracy in bit-rate," in *Proc. of DCC*, March 2000, pp. 343–352.

[5] M. Gastpar and M. Vetterli, "On the capacity of wireless networks: The relay case," in *Proceedings of IEEE Infocom*, June 2002.

[6] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing*, Prentice Hall, Englewood Cliffs, New Jersey, 1989.

[7] F. Marvasti, "Nonuniform sampling," in *Advanced Topics in Shannon Sampling and Interpolation*, R. J. Marks II, Ed. Springer-Verlag, 1993.

[8] F. J. Beutler, "Error-free reconvery of signals from irregularly spaced samples," *SIAM Review*, vol. 8, no. 3, pp. 328–335, 1966.

[9] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, U.K., 1985.

[10] D. Slepian and J. Wolf, "Noiseless coding of correlated information sources," *Transactions on Information Theory*, vol. IT-19, pp. 471–480, July 1973.

[11] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, Wiley, New York, 1991.

[12] I. E. Telatar, "Capacity of multi-antenna Gaussian channels," *European Trans. On Telecommunications*, vol. 10, pp. 585–589, Nov/Dec 1999.