A Theoretical Look at Pixel Ordering

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Overview

1. The Problem
2. The Badness Measure
3. Raster Scan Pixel Ordering
4. The Multi-Level Progressive (MLP) Method
5. The Problem of Weights

1. The Problem

Predictive Coding

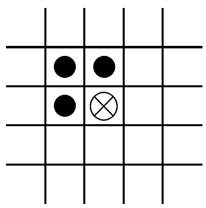
For every pixel P

1. **Prediction** - Predict pixel P using some previously processed pixels (neigbors).

2. **Error Modelling** - Obtain the probability of the prediction error.

3. **Coding** - Encode the prediction error using the probability obtained.

Example of Predictive Coding



Neighbors

(0,0)

31	32	6	72	35	12	24	8
35	29	11	54	32	21	27	97
24	38	54	55	31	27	22	70
6	31	52	55	31	31	6	54
9	31	56	27	22	29	57	63
88	35	62	24	4	30	54	71
75	34	18	20	5	31	52	24
55	32	24	19	94	33	63	23

8 x 8 image

Using the mean as the prediction function,

Predicted value $\hat{Y}_{(1,1)} = \frac{1}{3}(31 + 32 + 35) = 32$

Since actual value $Y_{(1,1)} = 29$,

Prediction error $= \hat{Y}_{(1,1)} - Y_{(1,1)} = 3$

The Problem

- *Pixel ordering* is the order or sequence in which pixels are processed or encoded.
- *Pixel choice* is the selection of neighbors used for prediction.
- Pixel choice is constrained by pixel ordering.

This thesis investigates the effect of pixel ordering and pixel choice on the code length of prediction errors, i.e.,

What pixel ordering and pixel choice give good compression?

Preliminaries

n The total no. of pixels in an image.

s The total no. of unpredictable starting pixels.

c The total no. of neighbors used in a prediction.

neighbors The pixels used to predict the current pixel.

$X_1, X_2, \ldots X_c$ Random variables for the neighbors used in a prediction.

$\hat{Y}$ The predicted value.

Y The true value.

error The difference between the predicted and the true value.

2. The Badness Measure

The Badness Measure

$$\text{Badness } \mathcal{B} = \sum_{\text{all pixels}} \begin{cases} z' & \text{no prediction} \\ \dfrac{\sqrt{\sum_d c_d f(d)}}{\sum_d c_d} & \text{otherwise} \end{cases}$$

d	Manhattan distances of neighbors.
c_d	no. of distance d neighbors.
$f(d)$	a function that increases with increasing distance.
z'	the cost of encoding a pixel instead of its prediction error.

Derivation of $\mathcal{B}$ (i)

- The entropy H or average bits per pixel is

$$H = -\sum^{\text{all error value i}} P(X=i)\log P(X=i)$$

$$= -\sum_{\text{all } n \text{ errors}} \left(\frac{1}{n}\right) \log_2 P(\textit{error})$$

- Code length of a prediction error is

$$-\log_2 P(\textit{error})$$

- Prediction errors have zero mean and follow the Laplacian distribution

$$f_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\sigma^2}} \exp\left(-\sqrt{\frac{2}{\sigma^2}}|x-\mu|\right)$$

- Code length $\propto$ $|\textit{error}|$

- Define $\mathcal{B} \approx |\textit{error}|$

Derivation of $\mathcal{B}$ (ii)

$$|error| = \sqrt{E\left[\left(\hat{Y} - Y\right)^2\right]}. \tag{1}$$

But $E(\hat{Y} - Y) = 0$ implies $Y = E(\hat{Y})$,

$$|error| = \sqrt{E\left[\left(\hat{Y} - E(\hat{Y})\right)^2\right]}. \tag{2}$$

$$|error| = \sqrt{Var\left(\hat{Y}\right)} \tag{3}$$

$$= \sqrt{Var\left(\hat{Y} - Y\right)}. \tag{4}$$

Derivation of $\mathcal{B}$ (iii)

Suppose $X_1, X_2, \ldots X_c$ are the random variables for the pixel values of the c neighbors used in an arbitrary prediction.

$$error = \hat{Y} - Y \tag{5}$$

$$\approx \frac{1}{c} \sum_{i=1}^{c} \left(X_i - Y \right). \tag{6}$$

Derivation of $\mathcal{B}$ (iv)

Substituting into our previous expression for $|error|$,

$$|error| = \sqrt{Var\left[\frac{1}{c}\sum_{i=1}^{c}(X_i - Y)\right]}. \tag{7}$$

Assuming $(X_i - Y)$'s are independent,

$$|error| = \sqrt{\frac{1}{c^2}\sum_{i=1}^{c} Var\left(X_i - Y\right)} \tag{8}$$

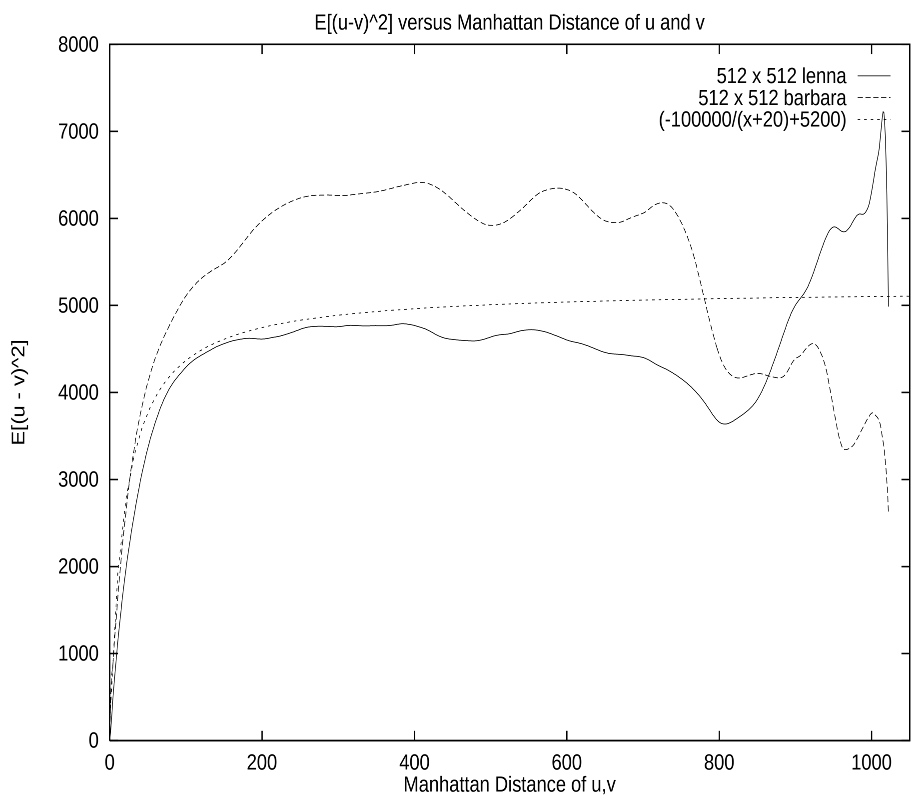
$$= \frac{1}{c}\sqrt{\sum_{i=1}^{c} Var\left(X_i - Y\right)} \tag{9}$$

$$= \frac{\sqrt{\sum_d c_d f(d)}}{\sum_d c_d}. \tag{10}$$

where $f(d)$ returns $Var\left(X_i - Y\right)$ given the distance d between X_i and Y.

Estimating $f(d)$

- $f(d)$ associated with $Var\,(U - V)$.
- Experimentally obtain for each d the $Var\,(U - V)$ over all distance d pixel pairs (U, V) for some images.



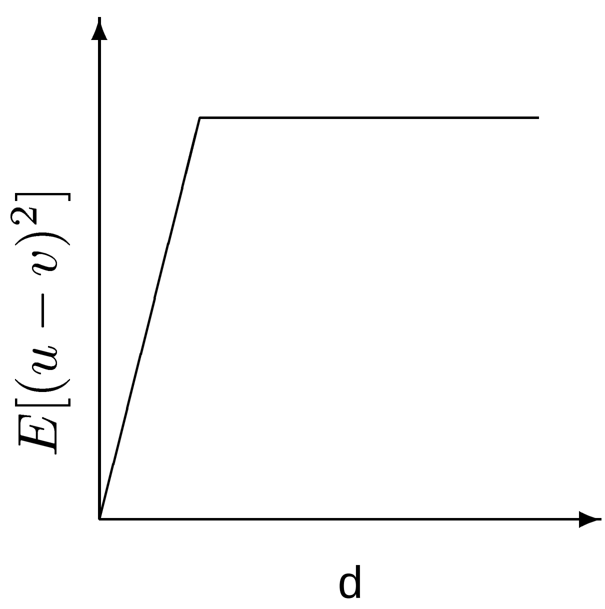
- The plot resembles $f(d) = a - \frac{b}{d+c}$ for many natural images.

Linear Approximation

- Using 2 straight lines piecewise.
- Definition:

$$f(d) = \begin{cases} ud & \text{if } 0 < d \leq \alpha, \\ f_{\text{max}} & \text{if } d > \alpha. \end{cases}$$

- Graph:

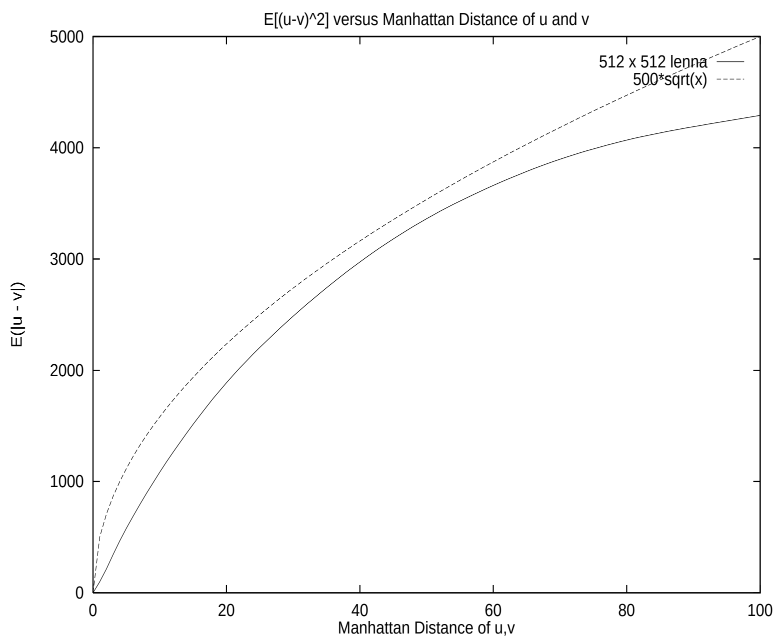


Sublinear Approximation

- Using 1 square root curve and 1 straight line piecewise.
- Definition:

$$f(d) = \begin{cases} u\sqrt{d} & \text{if } 0 < d \leq \alpha, \\ f_{\text{max}} & \text{if } d > \alpha. \end{cases}$$

- Plot of $E[(u-v)^2]$ for lenna and $500\sqrt{d}$:

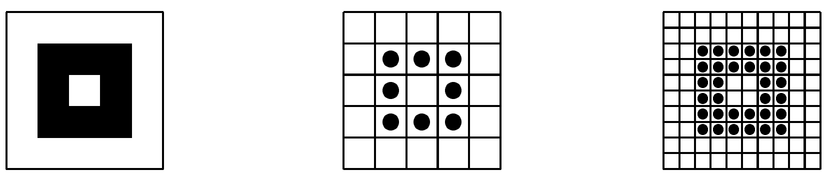


Some Properties of $f(d)$

- Resolution of image

$$u = \frac{w}{\sqrt{n}}$$

where w is some positive constant, not dependent upon n.

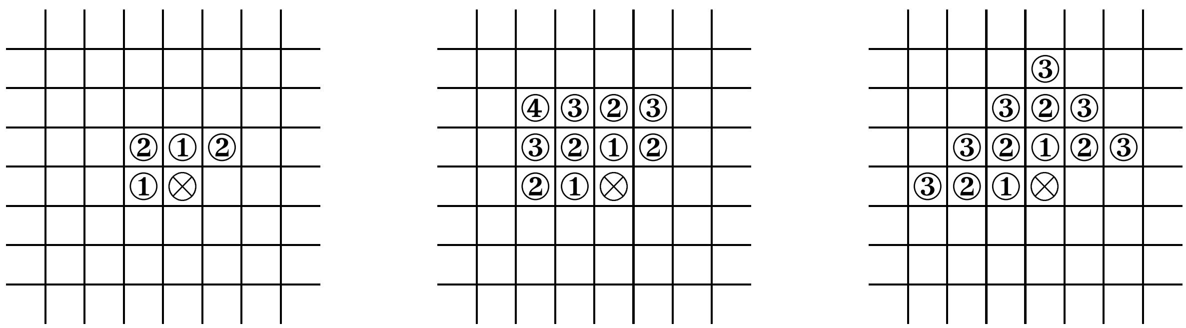


The real picture 5×5 image 10×10 image

- Neighbors at huge distances seldom used.

3. Raster Scan Pixel Ordering

Raster Scan Pixel Ordering



A **B** **C**

- The no. unpredictable starting pixels s depends on $\sqrt{n}$.
- But s can be minimized by applying predictive techniques.

Analysis of Raster Scan Pixel Ordering

Using our proposed badness measure,

$$\mathcal{B}_{rasterscan} = sz' + \sum_{p=1}^{n-s} \frac{\sqrt{\sum_{d=1}^{l} c_d f(d)}}{\sum_{d=1}^{l} c_d}. \quad (11)$$

For rasterscan, $c_d \leq 2d$; using the maximum c_d,

$$\mathcal{B}_{rasterscan} = sz' + \sum_{p=1}^{n-s} \frac{\sqrt{\sum_{d=1}^{l} (2d) f(d)}}{\sum_{d=1}^{l} 2d}. \quad (12)$$

Analysis of Raster Scan using $f(d) = ud$ **(i)**

Substituting $f(d) = ud$,

$$\mathcal{B}_{rasterscan} = sz' + \sum_{p=1}^{n-s} \frac{\sqrt{\sum_{d=1}^{l} (2d)(ud)}}{2\sum_{d=1}^{l} d} \qquad (13)$$

$$= sz' + \sum_{p=1}^{n-s} \frac{\sqrt{2u\sum_{d=1}^{l} d^2}}{2\sum_{d=1}^{l} d} \qquad (14)$$

$$= sz' + \sum_{p=1}^{n-s} \frac{\sqrt{u\sum_{d=1}^{l} d^2}}{\sqrt{2}\sum_{d=1}^{l} d}. \qquad (15)$$

Analysis of Raster Scan using $f(d) = ud$ (ii)

Using the identities $\sum_{d=1}^{l} d^2 = \frac{l}{6}(l+1)(2l+1)$ and $\sum_{d=1}^{l} d = \frac{l}{2}(l+1)$,

$$\mathcal{B}_{rasterscan} = sz' + \sum_{p=1}^{n-s} \frac{\sqrt{u\left[\frac{l}{6}(l+1)(2l+1)\right]}}{\sqrt{2}\left[\frac{l}{2}(l+1)\right]} \quad (16)$$

$$= sz' + \sum_{p=1}^{n-s} \sqrt{\frac{u(2l+1)}{3l(l+1)}} \quad (17)$$

$$= sz' + (n-s)\sqrt{\frac{u(2l+1)}{3l(l+1)}}. \quad (18)$$

Analysis of Raster Scan using $f(d) = ud$ (cont' iii)

Substituting $u = \frac{w}{\sqrt{n}}$

$$\mathcal{B}_{rasterscan} = sz' + (n - s)\sqrt{\frac{w}{\sqrt{n}}\left[\frac{(2l+1)}{3l(l+1)}\right]} \tag{19}$$

$$= sz' + \left(n^{\frac{3}{4}} - sn^{-\frac{1}{4}}\right)\sqrt{\frac{w(2l+1)}{3l(l+1)}}. \tag{20}$$

Analysis of Raster Scan using $f(d) = u\sqrt{d}$ (i)

Substituting $f(d) = u\sqrt{d}$,

$$= sz' + \sum_{p=1}^{n-s} \frac{\sqrt{\sum_{d=1}^{l}(2d)(u\sqrt{d})}}{2\sum_{d=1}^{l} d} \tag{21}$$

$$= sz' + \sum_{p=1}^{n-s} \frac{\sqrt{2u\sum_{d=1}^{l} d^{\frac{3}{2}}}}{2\sum_{d=1}^{l} d} \tag{22}$$

$$= sz' + \sum_{p=1}^{n-s} \frac{\sqrt{u\sum_{d=1}^{l} d^{\frac{3}{2}}}}{\sqrt{2}\sum_{d=1}^{l} d}. \tag{23}$$

Analysis of Raster Scan using $f(d) = u\sqrt{d}$ (ii)

Using the identity $\sum_{d=1}^{l} d = \frac{l}{2}(l+1)$,

$\mathcal{B}_{rasterscan}$

$$= sz' + \sum_{p=1}^{n-s} \frac{\sqrt{u\sum_{d=1}^{l} d^{\frac{3}{2}}}}{\sqrt{2}\left[\frac{l}{2}(l+1)\right]} \tag{24}$$

$$= sz' + \sum_{p=1}^{n-s} \frac{\sqrt{2u\sum_{d=1}^{l} d^{\frac{3}{2}}}}{l(l+1)}. \tag{25}$$

Using the approximation
$\sum_{i=1}^{l} i^{\frac{3}{2}} \approx \int_{0.5}^{l+0.5} i^{\frac{3}{2}} di = \frac{2}{5}(l+0.5)^{\frac{5}{2}} - \frac{2}{5}(0.5)^{\frac{5}{2}}$,

$$= sz' + \sum_{p=1}^{n-s} \frac{\sqrt{2u\left[\frac{2}{5}\left(l+0.5\right)^{\frac{5}{2}} - \frac{2}{5}\left(0.5\right)^{\frac{5}{2}}\right]}}{l(l+1)}. \tag{26}$$

Analysis of Raster Scan using $f(d) = u\sqrt{d}$ (iii)

Substituting $u = \frac{w}{\sqrt[4]{n}}$,

$\mathcal{B}_{rasterscan}$

$$= sz' + \sum_{p=1}^{n-s} \frac{\sqrt{2\frac{w}{\sqrt[4]{n}}\left[\frac{2}{5}\left(l+0.5\right)^{\frac{5}{2}} - \frac{2}{5}\left(0.5\right)^{\frac{5}{2}}\right]}}{l(l+1)} \tag{27}$$

$$= sz' + (n-s)\frac{n^{-\frac{1}{8}}\sqrt{2w\left[\frac{2}{5}\left(l+0.5\right)^{\frac{5}{2}} - \frac{2}{5}\left(0.5\right)^{\frac{5}{2}}\right]}}{l(l+1)}. \tag{28}$$

$$= \left\{\frac{\sqrt{2w\left[\frac{2}{5}\left(l+0.5\right)^{\frac{5}{2}} - \frac{2}{5}\left(0.5\right)^{\frac{5}{2}}\right]}}{l(l+1)}\right\}n^{\frac{7}{8}} + sz' - \left\{\frac{\sqrt{2w\left[\frac{2}{5}\left(l+0.5\right)^{\frac{5}{2}} - \frac{2}{5}\left(0.5\right)^{\frac{5}{2}}\right]}}{l(l+1)}\right\}n^{-\frac{1}{8}}. \tag{29}$$

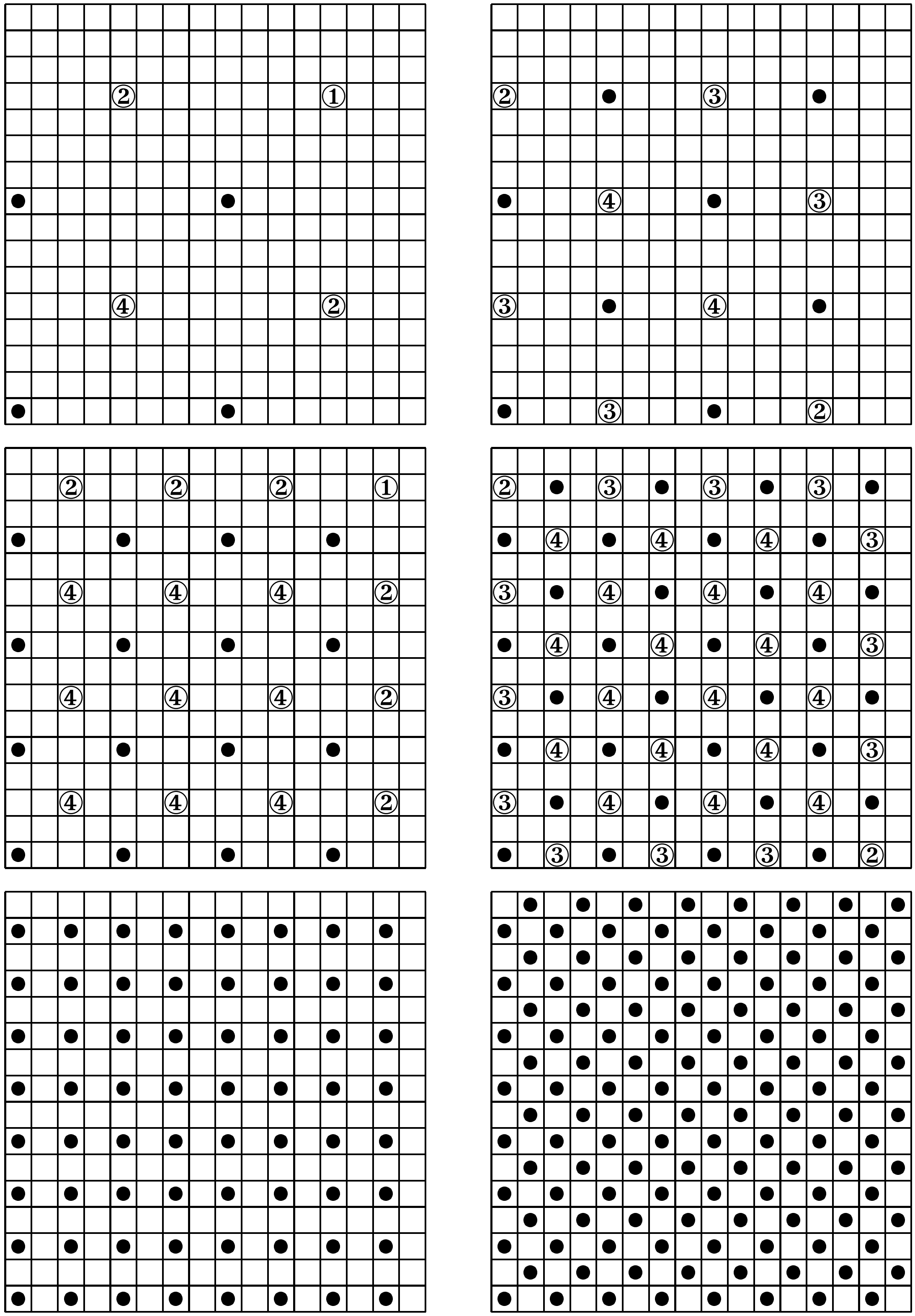
Badness of Raster Scan

l	$f(d) = ud$	$f(d) = u\sqrt{d}$
1	$0.707\sqrt{w}n^{\frac{3}{4}}$	$0.718\sqrt{w}n^{\frac{7}{8}}$
2	$0.527\sqrt{w}n^{\frac{3}{4}}$	$0.464\sqrt{w}n^{\frac{7}{8}}$
3	$0.441\sqrt{w}n^{\frac{3}{4}}$	$0.355\sqrt{w}n^{\frac{7}{8}}$
4	$0.387\sqrt{w}n^{\frac{3}{4}}$	$0.293\sqrt{w}n^{\frac{7}{8}}$
5	$0.350\sqrt{w}n^{\frac{3}{4}}$	$0.251\sqrt{w}n^{\frac{7}{8}}$
6	$0.321\sqrt{w}n^{\frac{3}{4}}$	$0.221\sqrt{w}n^{\frac{7}{8}}$
7	$0.300\sqrt{w}n^{\frac{3}{4}}$	$0.198\sqrt{w}n^{\frac{7}{8}}$

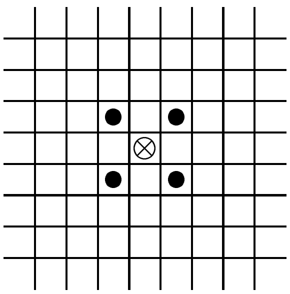
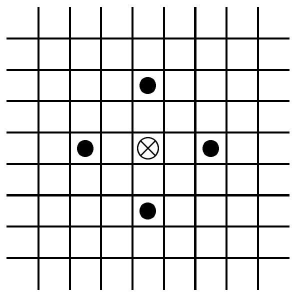
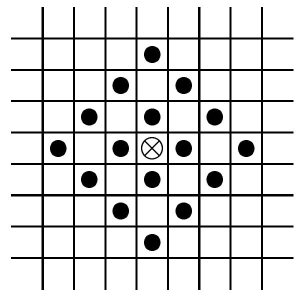
Table of Dominant Term of $\mathcal{B}_{rasterscan}$

4. The Multi-Level Progressive (MLP) Method

An Example of MLP Pixel Ordering



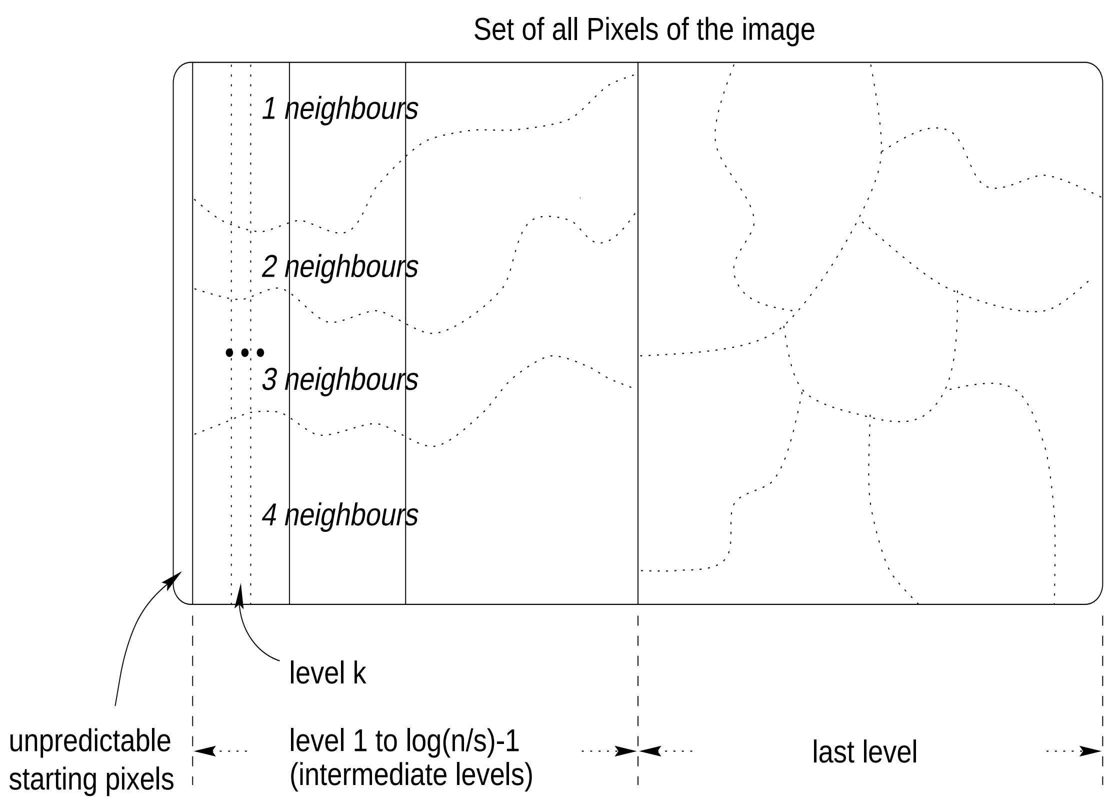
The MLP Method

Intermediate Levels		Last Level
odd levels	even levels	
		

- Distance of neighbors at different intermediate levels are different.
- Distance of neighbors at the same intermediate levels are the same.
- Distance of neighbors are halved every intermediate level.
- No. of predictions doubles every level.
- An arbitrary pixel is either a unpredictable starting pixel OR predicted exactly once at exactly 1 level.

Analyzing MLP (i)

- Partition all predictions into those occuring in intermediate levels or last level.
- Partition all predictions of a particular level according to no. of neighbors used.
- Find the cardinality of these partitions.



Analyzing MLP (ii)

The number of pixels predicted at level k can be formulated recursively as,

$$n_k = \begin{cases} s & \text{if } k = 1 \\ 2n_{k-1} & \text{otherwise.} \end{cases} \quad (30)$$

$$= 2^{k-1}s. \quad (31)$$

The total number of levels is,

$$\begin{aligned} n_{last} &= \frac{n}{2} \\ 2^{last-1}s &= \frac{n}{2} \\ 2^{last} &= \frac{n}{s} \\ last &= \log_2\left(\frac{n}{s}\right). \end{aligned} \quad (32)$$

Since n and s are assumed to be perfect squares in base 2, $last$ is even.

Analyzing MLP (iii)

Since n and s are assumed to be perfect squares and in base 2, the distance of the neighbors at the start of the first level can be expressed as,

$$d_1 = \sqrt{\frac{n}{s}} \tag{33}$$

and the distance of the neighbors used in the prediction at level k (except last level) as,

$$d_k = \begin{cases} \sqrt{\frac{n}{s}} & \text{if } k = 1, \\ d_{k-1} & \text{if } k \text{ is odd} \\ \frac{1}{2} d_{k-1} & \text{if } k \text{ is even} \end{cases} \tag{34}$$

$$= 2^{-\lfloor k/2 \rfloor} \sqrt{\frac{n}{s}}. \tag{35}$$

Analyzing MLP (iv)

Cardinality of the c-neighbors subpartition in the kth-level partition (k is not the last level).

c	$n_{k,c}$ for odd k	$n_{k,c}$ for even k
1	1	0
2	$2(\sqrt{n_k} - 1)$	2
3	0	$4(\sqrt{n_{k-1}} - 1)$
4	$(\sqrt{n_k} - 1)^2$	$(\sqrt{n_{k-1}} - 1)(2\sqrt{n_{k-1}} - 2)$

Analyzing MLP (v)

Cardinality of the c-neighbors subpartition inside the last level partition.

$c = \sum_d c_d$	c_1	c_3	$n_{k,c}$ for even last level k
6	2	4	2
8	3	5	4
9	3	6	4
10	3	7	$2 + 4\left(\frac{\sqrt{n}-6}{2}\right)$
12	4	8	4
13	4	9	$4\left(\frac{\sqrt{n}-6}{2}\right)$
14	4	10	2
15	4	11	$4\left(\frac{\sqrt{n}-6}{2}\right)$
16	4	12	$(\sqrt{n} - 6)\left(\frac{\sqrt{n}-6}{2}\right)$
otherwise			0

Analyzing of MLP (vi)

$$\begin{aligned}\mathcal{B}_{MLP} &= sz' + \mathcal{B}_{intermediate} + \mathcal{B}_{last} \\ &= sz' + \sum_{k=1}^{m} \left\{ \sum_{c=1}^{4} n_{k,c} \frac{\sqrt{\sum_d c_d f(d)}}{c} \right\} \\ &\quad + \sum_{c=1}^{16} n_{last,c} \frac{\sqrt{\sum_d c_d f(d)}}{c}.\end{aligned}$$

where

m is the number of intermediate levels and

$n_{k,c}$ the number of pixels in the partition representing predictions in level k and with c neighbors.

Analysis of MLP Intermediate Levels (i)

Summing from an arbitrary level v to m,

$$\mathcal{B}_{intermediate} = \sum_{k=v}^{m} \left[\sum_{c=1}^{4} n_{k,c} \frac{\sqrt{cf(d_k)}}{c} \right] \tag{36}$$

$$= \sum_{k=v}^{m} \left[\sum_{c=1}^{4} n_{k,c} \sqrt{\frac{f(d_k)}{c}} \right] \tag{37}$$

$$= \sum_{k=v}^{m} \Bigg[n_{k,1} \sqrt{\frac{f(d_k)}{1}} + n_{k,2} \sqrt{\frac{f(d_k)}{2}} + n_{k,3} \sqrt{\frac{f(d_k)}{3}} + n_{k,4} \sqrt{\frac{f(d_k)}{4}} \Bigg]. \tag{38}$$

Analysis of MLP Intermediate Levels (ii)

Splitting the summation into odd and even k

$$\mathcal{B}_{intermediate} = \sum_{\substack{k=v \\ k \text{ odd}}}^{m} \sqrt{f(d_k)}\Big[n_{k,1} + \frac{1}{\sqrt{2}}n_{k,2} + \frac{1}{\sqrt{3}}n_{k,3} + \frac{1}{2}n_{k,4}\Big] + \sum_{\substack{k=v \\ k \text{ even}}}^{m} \sqrt{f(d_k)}\Big[n_{k,1} + \frac{1}{\sqrt{2}}n_{k,2} + \frac{1}{\sqrt{3}}n_{k,3} + \frac{1}{2}n_{k,4}\Big]. \tag{39}$$

Analysis of MLP Intermediate Levels (iii)

Substituting the expressions for $n_{k,c}$,

$$\mathcal{B}_{intermediate} = \sum_{\substack{k=v \\ k \text{ odd}}}^{m} \sqrt{f(d_k)} \Big[1 + \frac{2}{\sqrt{2}} \left(\sqrt{n_k} - 1\right) + \frac{1}{2} \left(\sqrt{n_k} - 1\right)^2 \Big]$$

$$+ \sum_{\substack{k=v \\ k \text{ even}}}^{m} \sqrt{f(d_k)} \Big[\frac{2}{\sqrt{2}} + \frac{4}{\sqrt{3}} \left(\sqrt{n_{k-1}} - 1\right) + \frac{1}{2} \left(\sqrt{n_{k-1}} - 1\right) \left(2\sqrt{n_{k-1}} - 2\right) \Big]. \quad (40)$$

Analysis of MLP Intermediate Levels (iv)

Grouping the n-terms together,

$$\mathcal{B}_{intermediate}$$

$$= \sum_{\substack{k=v \\ k \text{ odd}}}^{m} \sqrt{f(d_k)} \Big[\frac{1}{2} n_k + \left(\sqrt{2} - 1\right) \sqrt{n_k}$$

$$+ \left(\frac{3}{2} - \sqrt{2} \right) \Big]$$

$$+ \sum_{\substack{k=v \\ k \text{ even}}}^{m} \sqrt{f(d_k)} \Big[n_{k-1} + \left(\frac{4}{\sqrt{3}} - 2 \right) \sqrt{n_{k-1}}$$

$$+ \left(\sqrt{2} - \frac{4}{\sqrt{3}} + 1 \right) \Big]. \tag{41}$$

Analysis of MLP Intermediate Levels (v)

Substituting $n_k = 2^{k-1}s$,

$$\mathcal{B}_{intermediate}$$

$$= \sum_{\substack{k=v \\ k \text{ odd}}}^{m} \sqrt{f(d_k)} \Big[\frac{1}{4} 2^k s + \left(1 - \frac{1}{\sqrt{2}}\right) \sqrt{s} 2^{k/2} + \left(\frac{3}{2} - \sqrt{2}\right) \Big]$$

$$+ \sum_{\substack{k=v \\ k \text{ even}}}^{m} \sqrt{f(d_k)} \Big[\frac{1}{4} 2^k s + \left(\frac{2}{\sqrt{3}} - 1\right) \sqrt{s} 2^{k/2} + \left(\sqrt{2} - \frac{4}{\sqrt{3}} + 1\right) \Big]. \tag{42}$$

Analysis of MLP Intermediate Levels (vi)

Substituting $f(d_k) = ud_k$ and $d_k = 2^{-\lfloor k/2 \rfloor}\sqrt{\frac{n}{s}}$,

$$\mathcal{B}_{intermediate}$$

$$= \sum_{\substack{k=v \\ k \text{ odd}}}^{m} \sqrt{u 2^{-\lfloor k/2 \rfloor}\sqrt{\frac{n}{s}}}\Big[\frac{1}{4}2^k s + \left(1 - \frac{1}{\sqrt{2}}\right)\sqrt{s}2^{k/2} + \left(\frac{3}{2} - \sqrt{2}\right)\Big]$$

$$+ \sum_{\substack{k=v \\ k \text{ even}}}^{m} \sqrt{u 2^{-\lfloor k/2 \rfloor}\sqrt{\frac{n}{s}}}\Big[\frac{1}{4}2^k s + \left(\frac{2}{\sqrt{3}} - 1\right)\sqrt{s}2^{k/2} + \left(\sqrt{2} - \frac{4}{\sqrt{3}} + 1\right)\Big]. \quad (43)$$

Analysis of MLP Intermediate Levels (vii)

If k is even, $\lfloor k/2 \rfloor = k/2$, else $\lfloor k/2 \rfloor = (k-1)/2$,

$$\mathcal{B}_{intermediate}$$

$$= \sum_{\substack{k=v \\ k \text{ odd}}}^{m} \sqrt{u 2^{-(k-1)/2} \sqrt{\frac{n}{s}} \Big[\frac{1}{4} 2^k s + \left(1 - \frac{1}{\sqrt{2}}\right) \sqrt{s} 2^{k/2} + \left(\frac{3}{2} - \sqrt{2}\right) \Big]}$$

$$+ \sum_{\substack{k=v \\ k \text{ even}}}^{m} \sqrt{u 2^{-k/2} \sqrt{\frac{n}{s}} \Big[\frac{1}{4} 2^k s + \left(\frac{2}{\sqrt{3}} - 1\right) \sqrt{s} 2^{k/2} + \left(\sqrt{2} - \frac{4}{\sqrt{3}} + 1\right) \Big]}. \quad (44)$$

Analysis of MLP Intermediate Levels (viii)

Since $last$ is even, $m = last - 1$ is odd, we bring out the '$k = m$'-term from the summation over odd k's,

$\mathcal{B}_{intermediate}$

$$= \sum_{\substack{k=v \\ k \text{ odd}}}^{m-1} \Bigg\{ \sqrt{u2^{-(k-1)/2}\sqrt{\frac{n}{s}}} \Big[\frac{1}{4}2^k s + \left(1 - \frac{1}{\sqrt{2}}\right)\sqrt{s}2^{k/2} + \left(\frac{3}{2} - \sqrt{2}\right) \Big] + \sqrt{u2^{-(m-1)/2}\sqrt{\frac{n}{s}}} \Big[\frac{1}{4}2^m s + \left(1 - \frac{1}{\sqrt{2}}\right)\sqrt{s}2^{m/2} + \left(\frac{3}{2} - \sqrt{2}\right) \Big] \Bigg\}$$

$$+ \sum_{\substack{k=v \\ k \text{ even}}}^{m-1} \sqrt{u2^{-k/2}\sqrt{\frac{n}{s}}} \Big[\frac{1}{4}2^k s + \left(\frac{2}{\sqrt{3}} - 1\right)\sqrt{s}2^{k/2} + \left(\sqrt{2} - \frac{4}{\sqrt{3}} + 1\right) \Big]. \tag{45}$$

Analysis of MLP Intermediate Levels (ix)

Change indexing variable $k \to j$, where odd $k = v + 2j - 2$ and even $k = v + 2j - 1$, assuming v is odd,

$$\begin{aligned}
&\mathcal{B}_{intermediate} \\
&\quad = \sum_{j=1}^{\frac{m-v}{2}} \Bigg\{ \sqrt{u 2^{-\frac{v+2j-3}{2}} \sqrt{\frac{n}{s}}} \Big[\frac{1}{4} 2^{v+2j-2} s \\
&\qquad\qquad + \left(1 - \frac{1}{\sqrt{2}}\right) \sqrt{s} 2^{\frac{v+2j-2}{2}} + \left(\frac{3}{2} - \sqrt{2}\right) \Big] \Bigg\} \\
&\qquad + \sum_{j=1}^{\frac{m-v}{2}} \Bigg\{ \sqrt{u 2^{-\frac{v+2j-1}{2}} \sqrt{\frac{n}{s}}} \Big[\frac{1}{4} 2^{v+2j-1} s \\
&\qquad\qquad + \left(\frac{2}{\sqrt{3}} - 1\right) \sqrt{s} 2^{\frac{v+2j-1}{2}} + \left(\sqrt{2} - \frac{4}{\sqrt{3}} + 1\right) \Big] \Bigg\} \\
&\qquad + \sqrt{u 2^{(1-m)/2} \sqrt{\frac{n}{s}}} \Big[\frac{1}{4} 2^{m} s + \left(1 - \frac{1}{\sqrt{2}}\right) \sqrt{s} 2^{m/2} \\
&\qquad\qquad + \left(\frac{3}{2} - \sqrt{2}\right) \Big]. \qquad (46)
\end{aligned}$$

Analysis of MLP Intermediate Levels (x)

Simplifying and grouping j-terms together,

$$\mathcal{B}_{intermediate}$$

$$= \sqrt{u2^{\frac{1-v}{2}}\sqrt{\frac{n}{s}}} \sum_{j=1}^{\frac{m-v}{2}} \Bigg\{ \left[\frac{1}{4\sqrt{8}} + \frac{1}{8}\right] s2^{v}2^{\frac{3}{2}j}$$

$$+ \left[\sqrt{\frac{2}{3}} - \frac{1}{2}\right] \sqrt{s}2^{\frac{v}{2}}2^{\frac{j}{2}}$$

$$+ \left[\frac{5}{\sqrt{2}} - \frac{4}{\sqrt{3}} - 1\right] 2^{-\frac{j}{2}} \Bigg\}$$

$$+ \sqrt{u2^{(1-m)/2}\sqrt{\frac{n}{s}}} \Bigg\{ \frac{1}{4}2^{m}s + \left[1 - \frac{1}{\sqrt{2}}\right] \sqrt{s}2^{m/2}$$

$$+ \left[\frac{3}{2} - \sqrt{2}\right] \Bigg\} \qquad (47)$$

Analysis of MLP Intermediate Levels (xi)

Replacing the constant expressions in square brackets with a_1, $a_2 \ldots a_5$ respectively,

$$\mathcal{B}_{intermediate}$$

$$= \sqrt{u 2^{\frac{1-v}{2}} \sqrt{\frac{n}{s}} \Big\{ a_1 s 2^v \sum_{j=1}^{\frac{m-v}{2}} (2^{\frac{3}{2} j}) + a_2 \sqrt{s} 2^{\frac{v}{2}} \sum_{j=1}^{\frac{m-v}{2}} (2^{\frac{j}{2}}) + a_3 \sum_{j=1}^{\frac{m-v}{2}} (2^{-\frac{j}{2}}) \Big\}} + \sqrt{u 2^{(1-m)/2} \sqrt{\frac{n}{s}} \left[\frac{1}{4} 2^m s + a_4 \sqrt{s} 2^{\frac{m}{2}} + a_5 \right]} . \tag{48}$$

Analysis of MLP Intermediate Levels (xii)

Summing the geometric series,

$$\mathcal{B}_{intermediate}$$

$$= \sqrt{u 2^{\frac{1-v}{2}} \sqrt{\frac{n}{s}}} \Bigg\{ \quad s 2^{v} \left[\frac{a_1 2^{\frac{3}{2}}}{2^{\frac{3}{2}} - 1} \right] \left(2^{\frac{3}{4}(m-v)} - 1 \right)$$

$$+ \sqrt{s} 2^{\frac{v}{2}} \left[\frac{a_2 2^{\frac{1}{2}}}{2^{\frac{1}{2}} - 1} \right] \left(2^{\frac{1}{4}(m-v)} - 1 \right)$$

$$+ \left[\frac{a_3 2^{-\frac{1}{2}}}{1 - 2^{-\frac{1}{2}}} \right] \left(1 - 2^{-\frac{1}{4}(m-v)} \right) \Bigg\}$$

$$+ \sqrt{u 2^{(1-m)/2} \sqrt{\frac{n}{s}}} \left[\frac{1}{4} 2^{m} s + a_4 \sqrt{s} 2^{\frac{m}{2}} + a_5 \right]. \quad (49)$$

Analysis of MLP Intermediate Levels (xiii)

Replacing constant expressions in $[\ldots]$ again and simplifying,

$$
\begin{aligned}
\mathcal{B}_{intermediate}
&= \sqrt{u 2^{\frac{1-v}{2}} \sqrt{\frac{n}{s}}} \Bigg[\quad \left(s a_6 2^{\frac{1}{4}v} \right) 2^{\frac{3}{4}m} - s 2^{v} a_6 \\
&\qquad + \left(\sqrt{s} a_7 2^{\frac{1}{4}v} \right) 2^{\frac{1}{4}m} - \sqrt{s} 2^{\frac{v}{2}} a_7 \\
&\qquad + a_8 - \left(a_8 2^{\frac{1}{4}v} \right) 2^{-\frac{1}{4}m} \Bigg] \\
&\quad + \sqrt{u 2^{(1-m)/2} \sqrt{\frac{n}{s}}} \left[\frac{1}{4} 2^{m} s + a_4 \sqrt{s} 2^{\frac{m}{2}} + a_5 \right].
\end{aligned}
\tag{50}
$$

Analysis of MLP Intermediate Levels (xiv)

Substituting $m = \log_2\left(\frac{n}{s}\right) - 1$ and consolidating non-m-terms at the end,

$$
\begin{aligned}
&\mathcal{B}_{intermediate} \\
&= \sqrt{u 2^{\frac{1-v}{2}} \sqrt{\frac{n}{s}}} \Big[\quad \left(s a_6 2^{\frac{1}{4}v}\right) 2^{\frac{3}{4}\left(\log_2\left(\frac{n}{s}\right)-1\right)} \\
&\qquad + \left(\sqrt{s} a_7 2^{\frac{1}{4}v}\right) 2^{\frac{1}{4}\left(\log_2\left(\frac{n}{s}\right)-1\right)} \\
&\qquad - \left(a_8 2^{\frac{1}{4}v}\right) 2^{-\frac{1}{4}\left(\log_2\left(\frac{n}{s}\right)-1\right)} \\
&\qquad + \left(a_8 - \sqrt{s} 2^{\frac{v}{2}} a_7 - s 2^v a_6\right) \Big] \\
&\quad + \sqrt{u 2^{\left(1-\log_2\left(\frac{n}{s}\right)+1\right)/2} \sqrt{\frac{n}{s}}} \Big[\frac{1}{4} 2^{\log_2\left(\frac{n}{s}\right)-1} s \\
&\qquad + a_4 \sqrt{s} 2^{\frac{1}{2}\left(\log_2\left(\frac{n}{s}\right)-1\right)} + a_5 \Big].
\end{aligned}
\tag{51}
$$

Analysis of MLP Intermediate Levels (xv)

Finally,

$$
\begin{aligned}
&\mathcal{B}_{intermediate} \\
&= \sqrt{u} \Big\{ \quad \left(2^{-\frac{1}{2}} a_6 + \frac{\sqrt{2}}{8} \right) n + (a_7 + a_4)\, n^{\frac{1}{2}} \\
&\qquad + \left(a_8 - \sqrt{s} 2^{\frac{v}{2}} a_7 - s 2^v a_6 \right) \left(2^{\frac{1-v}{4}} s^{-\frac{1}{4}} \right) n^{\frac{1}{4}} \\
&\qquad + \left(2^{\frac{1}{2}} a_5 - 2^{\frac{1}{2}} a_8 \right) \Big\} \qquad (52)
\end{aligned}
$$

Analysis of MLP the Last Level (i)

$$\mathcal{B}_{last}$$

$$= \sum_{c=1}^{16} n_{last,c} \frac{\sqrt{\sum_d c_d f(d)}}{c}. \tag{53}$$

Substituting $f(d) = ud$,

$$= \sqrt{u} \sum_{c=1}^{16} n_{last,c} \frac{\sqrt{\sum_d c_d d}}{c}. \tag{54}$$

Substituting the expressions for $n_{k,c}$,

$$\begin{aligned}
&= \sqrt{u}\Bigg\{2 \times \frac{\sqrt{2+4\times 3}}{6} + 4 \times \frac{\sqrt{3+5\times 3}}{8} \\
&\quad + 4 \times \frac{\sqrt{3+6\times 3}}{9} + \left[2 + 4\left(\frac{\sqrt{n}-6}{2}\right)\right] \frac{\sqrt{3+7\times 3}}{10} \\
&\quad + 4 \times \frac{\sqrt{4+8\times 3}}{12} + 4\left(\frac{\sqrt{n}-6}{2}\right) \frac{\sqrt{4+9\times 3}}{13} \\
&\quad + 2 \times \frac{\sqrt{4+10\times 3}}{14} + 4\left(\frac{\sqrt{n}-6}{2}\right) \frac{\sqrt{4+11\times 3}}{15} \\
&\quad + (\sqrt{n}-6)\left(\frac{\sqrt{n}-6}{2}\right) \frac{\sqrt{4+12\times 3}}{16}\Bigg\}.
\end{aligned} \tag{55}$$

Analysis of MLP the Last Level (ii)

$$\mathcal{B}_{last} = \sqrt{u}\Bigg\{\left[\frac{\sqrt{10}}{16}\right]n + \Big[\frac{2}{5}\sqrt{6} + \frac{2}{13}\sqrt{31} + \frac{2}{15}\sqrt{37} - \frac{3}{4}\sqrt{10}\Big]n^{\frac{1}{2}} + \Big[\frac{\sqrt{14}}{3} + \frac{3}{2}\sqrt{2} + \frac{4}{9}\sqrt{21} + \frac{2}{5}\sqrt{6} - \frac{12}{5}\sqrt{6} + \frac{2}{3}\sqrt{7} - \frac{12}{13}\sqrt{31} + \frac{\sqrt{34}}{7} - \frac{12}{15}\sqrt{37} + \frac{9}{4}\sqrt{10}\Big]\Bigg\}. \quad (56)$$

Replacing the constant expressions in square brackets with a_9, a_{10} and a_{11} respectively,

$$\mathcal{B}_{last} = \sqrt{u}\Big[(a_9)\,n + (a_{10})\,n^{\frac{1}{2}} + (a_{11})\Big]. \quad (57)$$

The Badness of MLP using $f(d) = ud$

Substituting $\mathcal{B}_{intermediate}$ and $\mathcal{B}_{last}$ into the original equation for $\mathcal{B}_{MLP}$,

$$\begin{aligned}\mathcal{B}_{MLP} &\\ = sz' + \sqrt{u}\Bigg[&\left(2^{-\frac{1}{2}}a_6 + a_9 + \frac{\sqrt{2}}{8}\right)n \\ &+ (a_4 + a_7 + a_{10})\, n^{\frac{1}{2}} \\ &+ \left(a_8 - \sqrt{s}2^{\frac{v}{2}}a_7 - s2^{v}a_6\right)\left(2^{\frac{1-v}{4}}s^{-\frac{1}{4}}\right)n^{\frac{1}{4}} \\ &+ \left(a_{11} + 2^{\frac{1}{2}}a_5 - 2^{\frac{1}{2}}a_8\right)\Bigg]. \end{aligned} \tag{58}$$

Subsitituting $v = 1$ and $u = \frac{w}{\sqrt{n}}$, and simplifying,

$$\begin{aligned} = sz' + \sqrt{w}\Bigg[&\left(2^{-\frac{1}{2}}a_6 + a_9 + \frac{\sqrt{2}}{8}\right)n^{\frac{3}{4}} \\ &+ (a_4 + a_7 + a_{10})\, n^{\frac{1}{4}} \\ &+ \left(a_8 - \sqrt{s}2^{\frac{1}{2}}a_7 - 2sa_6\right)\left(s^{-\frac{1}{4}}\right) \\ &+ \left(a_{11} + 2^{\frac{1}{2}}a_5 - 2^{\frac{1}{2}}a_8\right)n^{-\frac{1}{4}}\Bigg]. \end{aligned} \tag{59}$$

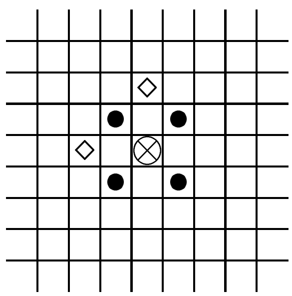
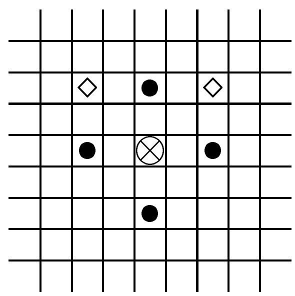
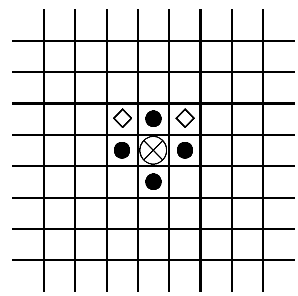
The Badness of MLP

Table of Dominant Term of $\mathcal{B}$

Pixel Ordering	$f(d) = ud$	$f(d) = u\sqrt{d}$
MLP	$0.608\sqrt{w}n^{\frac{3}{4}}$	$0.473\sqrt{w}n^{\frac{7}{8}}$
Raster Scan $l = 2$	$0.527\sqrt{w}n^{\frac{3}{4}}$	$0.464\sqrt{w}n^{\frac{7}{8}}$
Raster Scan $l = 3$	$0.44\sqrt{w}n^{\frac{3}{4}}$	$0.35\sqrt{w}n^{\frac{7}{8}}$

- How does MLP compare with Raster Scan?
- What if the no. of neighbors in both the intermediate and last level are fixed?
- What if more neighbors are used?

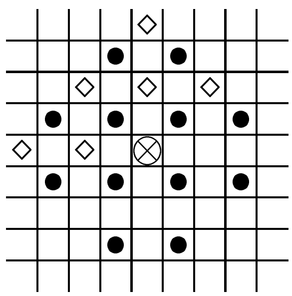
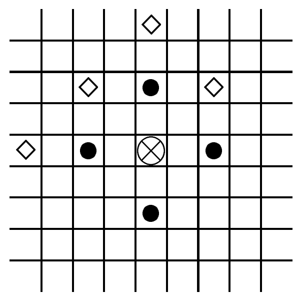
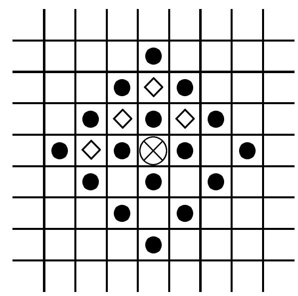
The Badness of GMLP6

Intermediate Levels		Last Level
odd levels	even levels	
		

- Dominant term of $\mathcal{B}_{GMLP6}$ and $\mathcal{B}_{rasterscan}$ (analyzed using $f(d) = ud$):

Raster Scan, $l = 3$	$0.441\sqrt{w}n^{\frac{3}{4}}$
GMLP6	$0.588\sqrt{w}n^{\frac{3}{4}}$

The Badness of GMLP20

Intermediate Levels		Last Level
odd levels	even levels	
		

- Dominant term of $\mathcal{B}_{GMLP6}$ and $\mathcal{B}_{rasterscan}$ (analyzed using $f(d) = ud$):

Raster Scan, $l = 3$	$0.441\sqrt{w}n^{\frac{3}{4}}$
GMLP20	$0.458\sqrt{w}n^{\frac{3}{4}}$
Raster Scan, $l = 2$	$0.527\sqrt{w}n^{\frac{3}{4}}$

- GMLP20 is probably better than most raster scan orderings using neighbors that are at most distance 2 away.

5. The Problem of Weights

What about weights?

- Pixel ordering not fully independent from prediction function.
- Neighbors with small weights can effectively be ignored.
- Our approach: a badness measure using optimal weights.

Derivation of Optimally Weighted $\mathcal{B}$ (i)

Let $X_1, X_2, \ldots, X_c$ be the random variables for the pixel values of the c neighbors used in an arbitrary prediction and $w_1, w_2, \ldots, w_c$ their corresponding normalized weights.

The random variable $\hat{Y}$ for the predicted value is then,

$$\hat{Y} = w_1 X_1 + w_2 X_2 + \ldots + w_c X_c. \quad (60)$$

If Y is the actual value of the pixel, the error will be given by,

$$\hat{Y} - Y = w_1 X_1 + w_2 X_2 + \ldots + w_c X_c - Y. \quad (61)$$

Since $\sum_i^c w_i = 1$, we distribute Y according to w_i,

$$\hat{Y} - Y = w_1(X_1 - Y) + w_2(X_2 - Y) + \ldots + w_c(X_c - Y). \quad (62)$$

Derivation of Optimally Weighted $\mathcal{B}$ (ii)

But we are interested in variances, so,

$$\text{Var}(\hat{Y} - Y) = \text{Var}\Big[w_1(X_1 - Y) + w_2(X_2 - Y) + \ldots + w_c(X_c - Y)\Big]. \tag{63}$$

Assuming $(X_i - Y)$ are independent,

$$\text{Var}(\hat{Y} - Y) = w_1^2 \text{Var}(X_1 - Y) + w_2^2 \text{Var}(X_2 - Y) + \ldots + w_c^2 \text{Var}(X_c - Y). \tag{64}$$

Estimating Var$(X_i - Y)$ with $f(d_i)$,

$$\text{Var}(\hat{Y} - Y) = w_1^2 f(d_1) + w_2^2 f(d_2) + \ldots + w_c^2 f(d_c). \tag{65}$$

Derivation of Optimally Weighted $\mathcal{B}$ (iii)

We use Lagrange's Method to obtain the optimal weights.

Let

$$F(\vec{w}) = w_1^2 f(d_1) + w_2^2 f(d_2) + \ldots + w_c^2 f(d_c) \tag{66}$$

$$G(\vec{w}) = w_1 + w_2 + \ldots + w_c = 1 \tag{67}$$

$$\begin{aligned} u &= F(\vec{w}) + \lambda G(\vec{w}) \\ &= w_1^2 f(d_1) + w_2^2 f(d_2) + \ldots + w_c^2 f(d_c) \\ &\quad + \lambda \left(w_1 + w_2 + \ldots + w_c \right) \end{aligned} \tag{68}$$

where $F(\vec{w})$ is the expression to be minimized under the constraint $G(\vec{w})$ and λ is the Lagrange multiplier.

Differentiating u partially with respect to each w_i,

$$\frac{\partial u}{\partial w_i} = 2 f(d_i) w_i + \lambda. \tag{69}$$

Derivation of Optimally Weighted $\mathcal{B}$ (iv)

At minima, each $\frac{\partial u}{\partial w_i}$ must be 0,

$$2f(d_i)w_i + \lambda = 0 \tag{70}$$

$$w_i = -\frac{\lambda}{2f(d_i)}. \tag{71}$$

Substituting into $G(\vec{w})$ to solve for λ,

$$\sum_{j=1}^{c} -\frac{\lambda}{2f(d_j)} = 1 \tag{72}$$

$$\frac{\lambda}{2}\sum_{j=1}^{c} \frac{1}{f(d_j)} = -1 \tag{73}$$

$$\lambda = -\frac{2}{\sum_{j=1}^{c} \frac{1}{f(d_j)}}. \tag{74}$$

Derivation of Optimally Weighted $\mathcal{B}$ (v)

Hence optimal weights are,

$$w_i = \Big(-\frac{1}{2f(d_i)}\Big)\Big(-\frac{2}{\sum_{j=1}^{c}\frac{1}{f(d_j)}}\Big) \tag{75}$$

$$= \frac{1}{f(d_i)\sum_{j=1}^{c}\frac{1}{f(d_j)}}. \tag{76}$$

Substituting into $F(\vec{w})$,

$$F(\vec{w}_{\text{optimal}}) = \sum_{i=1}^{c}\Bigg\{\Bigg[\frac{1}{f(d_i)\sum_{j=1}^{c}\frac{1}{f(d_j)}}\Bigg]^2 f(d_i)\Bigg\} \tag{77}$$

$$= \sum_{i=1}^{c}\frac{f(d_i)}{\left[f(d_i)\right]^2\left[\sum_{j=1}^{c}\frac{1}{f(d_j)}\right]^2}. \tag{78}$$

(79)

Derivation of Optimally Weighted $\mathcal{B}$ (vi)

But $\left[\sum_{j=1}^{c} \frac{1}{f(d_j)}\right]^2$ is constant relative to i,

$$F(\vec{w}_{\text{optimal}}) = \frac{1}{\left[\sum_{j=1}^{c} \frac{1}{f(d_j)}\right]^2} \sum_{i=1}^{c} \frac{f(d_i)}{[f(d_i)]^2} \quad (80)$$

$$= \frac{1}{\left[\sum_{j=1}^{c} \frac{1}{f(d_j)}\right]^2} \sum_{i=1}^{c} \frac{1}{f(d_i)} \quad (81)$$

$$= \frac{1}{\sum_{j=1}^{c} \frac{1}{f(d_j)}}. \quad (82)$$

The Optimal Weights Badness Measure $\mathcal{B}'$

$$\mathcal{B}' = \sum_{\text{all pixels}} \begin{cases} z' & \text{no prediction,} \\ \sqrt{F(\vec{w}_{\text{optimal.}})} & \text{otherwise} \end{cases} \tag{83}$$

$$= \sum_{\text{all pixels}} \begin{cases} z' & \text{no prediction,} \\ \sqrt{\frac{1}{\sum_{j=1}^{c} \frac{1}{f(d_j)}}} & \text{otherwise.} \end{cases} \tag{84}$$

Conclusion

Conclusion

- Summary of results:

	$f(d) = ud$	$f(d) = u\sqrt{d}$
$\mathcal{B}'_{rasterscan,l=3}$	$0.408\sqrt{w}n^{\frac{3}{4}}$	$0.348\sqrt{w}n^{\frac{7}{8}}$
$\mathcal{B}_{rasterscan,l=3}$	$0.441\sqrt{w}n^{\frac{3}{4}}$	$0.355\sqrt{w}n^{\frac{7}{8}}$
$\mathcal{B}_{GMLP20}$	$0.458\sqrt{w}n^{\frac{3}{4}}$	-
$\mathcal{B}'_{rasterscan,l=2}$	$0.500\sqrt{w}n^{\frac{3}{4}}$	$0.456\sqrt{w}n^{\frac{7}{8}}$
$\mathcal{B}_{rasterscan,l=2}$	$0.527\sqrt{w}n^{\frac{3}{4}}$	$0.464\sqrt{w}n^{\frac{7}{8}}$
$\mathcal{B}'_{MLP}$	$0.587\sqrt{w}n^{\frac{3}{4}}$	$0.469\sqrt{w}n^{\frac{7}{8}}$
$\mathcal{B}_{GMLP6}$	$0.588\sqrt{w}n^{\frac{3}{4}}$	-
$\mathcal{B}_{MLP}$	$0.608\sqrt{w}n^{\frac{3}{4}}$	$0.473\sqrt{w}n^{\frac{7}{8}}$

Empirical Data

- Table of zero-order entropy of prediction errors.

Image	0-order Entropy	
	MLP	Raster Scan ($l = 2$)
lenna 512	4.81044	4.78031
barbara 512	5.77607	5.76520
goldhill 512	5.31637	5.22996
fruits 512	3.05386	2.94682
mri 256	5.36377	5.26937

- In the experiment, the mean is used as the prediction function.