

# On the complexity of cutting plane proofs using split cuts

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October 19, 2009

## Abstract

We prove a monotone interpolation property for split cuts which, together with results from Pudlák (1997), implies that cutting-plane proofs which use split cuts (or, equivalently, mixed-integer rounding cuts or Gomory mixed-integer cuts) have exponential length in the worst case.

**Keywords:** cutting-plane proof, split cut, mixed-integer rounding, monotone interpolation

## 1 Introduction

The complexity of different types of cutting-plane proofs has been a much studied topic in recent years. Some well-known classes of cutting planes (linear inequalities satisfied by integral points in polyhedra) are Gomory-Chvátal cuts [13], split cuts [9], mixed-integer rounding (MIR) cuts [20], and lift-and-project cuts [1]. Let  $Ax \leq b$  be a system of linear inequalities with rational coefficients in  $n$  variables. In this paper, by an integral solution of  $Ax \leq b$  we mean one with all components integral; thus we only study pure integer programs (unless stated otherwise). A *Gomory-Chvátal* (GC) cutting plane (or cut) for  $Ax \leq b$  is a linear inequality  $cx \leq \lfloor d \rfloor$  where  $c$  is an integral vector, and  $cx \leq d$  is satisfied by all solutions of  $Ax \leq b$ . A *Gomory-Chvátal cutting-plane proof* of  $cx \leq d$  from  $Ax \leq b$  with *length*  $L$  is a sequence of inequalities  $a_i x \leq d_i$  ( $i = 1, \dots, L$ ) such that the last inequality in the sequence is  $cx \leq d$ , and for  $i = 1, \dots, L$ , the inequality  $a_i x \leq d_i$  is a Gomory-Chvátal cut for the previous inequalities in the sequence and the inequalities in  $Ax \leq b$ . Cutting-plane proofs were introduced in [5]. Any rational inequality satisfied by integral solutions of  $Ax \leq b$  has a GC cutting plane proof [13], [5].

Pudlák [21] proved, extending ideas in [4],[17], that GC cutting plane proofs of inequalities satisfied by integral points in polyhedra contained in  $[0, 1]^n$  have exponential length in the worst case. (For polyhedra not contained in  $[0, 1]^n$ , such a result trivially follows from the example of Bondy given in the next paragraph.) A similar result for lift-and-project cutting plane proofs can be found in [10]. An inequality  $cx \leq d$  is a *lift-and-project* cut for  $P = \{x \in [0, 1]^n : Ax \leq b\}$  if for some index  $j$ ,  $cx \leq d$  is satisfied by points in  $P \cap \{x_j = 0\}$  and  $P \cap \{x_j = 1\}$ . A question left open in [10] is whether cutting-plane proofs using split cuts (or *split cut proofs*) have exponential worst-case complexity. An inequality  $cx \leq d$  is a *split cut* for  $P = \{x \in \mathbb{R}^n : Ax \leq b\}$  if  $cx \leq d$  is satisfied by points in  $P \cap \{\alpha x \leq \beta\}$  and  $P \cap \{\alpha x \geq \beta + 1\}$ , where  $\alpha, \beta$  are integral. We say that  $cx \leq d$  is derived from the *disjunction*  $\alpha x \leq \beta \vee \alpha x \geq \beta + 1$ . Each integral point in  $P$  satisfies every split cut for  $P$ .

Lift-and-project cuts and GC cuts are special cases of split cuts. Split cuts are equivalent to MIR cuts and Gomory mixed-integer (GMI) cuts [20]; these are currently the most important classes of cutting planes for general integer programs. Further, split cuts are substantially stronger than GC cuts and lift-and-project cuts. For the family of instances (from J. A. Bondy, given in [5])  $P_k = \{x, y : x \leq ky, x + ky \leq k\}$ ,  $k = 1, 2, \dots$ , the inequality  $x \leq 0$  is a split cut, but any GC cutting-plane proof has length at least  $k/2$ , which is exponential in the *encoding size* of  $P_k$  (the number of bits required to write the constraints of  $P_k$ ) which equals  $O(\log k)$ . The inequality  $\sum_{i=1}^n x_i \geq 1$  is a split cut for  $P = \{x \in [0, 1]^n : \sum_{i=1}^n x_i \geq 1/2\}$ , but any lift-and-project proof has length at least  $n$  [8].

We prove that split-cut proofs have exponential worst-case complexity, thus generalizing Pudlák's result [21] and our result [10]. Pudlák showed how to map a GC cutting-plane proof with length  $L$  of a specific inequality to a *monotone real circuit* (see Section 2) with  $O(\text{poly}(L))$  gates which tests for cliques of a certain size in an arbitrary graph. He separately proved that such monotone circuits have exponentially many gates. We show how to map a split cut proof of length  $L$  to a monotone circuit with  $O(\text{poly}(L))$  gates for the clique testing problem mentioned above (this is called *monotone interpolation*). This mapping is very similar to the one in [10] for lift-and-project cuts, and is based on a property of split cuts which we prove: if  $gx + hy \leq d$  is a split cut for  $\{x, y : Ax \leq e, By \leq f\}$ , then there exist numbers  $r$  and  $s$  with  $r + s \leq d$ , such that  $gx \leq r$  is implied by split cuts for  $Ax \leq e$ ,  $hy \leq s$  is implied by split cuts for  $By \leq f$ , and  $r$  and  $s$  can be computed from  $Ax \leq e$  and  $By \leq f$ , respectively, using monotone operations only. Our results imply that cutting-plane proofs which use GMI cuts, MIR cuts, two-step MIR cuts [11], or pairing inequalities [15] have exponential worst-case complexity.

Bonet, Pitassi and Raz [4] gave an exponential lower bound on the complexity of branch-and-cut proofs which use GC cuts, and branching on inequalities  $\alpha x \leq \beta$  and  $\alpha x \geq \beta + 1$ , where  $\alpha$  and  $\beta$  are integral, subject to the restriction that the GC cuts and the branching inequalities have polynomially bounded coefficients (see also [17], [16]). The results in this paper, combined with results in [10], imply that branch-and-cut proofs which use split cuts but branch only on the inequalities  $x_i \leq 0$  and  $x_i \geq 1$  for 0-1 variables  $x_i$ ,

have exponential worst-case complexity.

In the next section, we review well-known complexity results for monotone circuits. In Section 3, we present MIR cuts in the form given in [12], and discuss their equivalence with split cuts. Based on this equivalence, we prove a monotone interpolation result for split cuts in Section 4, and give our exponential lower bound result. The results in this paper are self-contained other than for Theorem 1 [21].

## 2 Monotone circuit complexity

Before defining monotone circuits, we first define the notion of a boolean circuit relevant to us. See [3] for a nice survey of complexity results on boolean circuits including the results of Razborov and of Alon and Boppana cited below. Let  $B$  stand for the set of elementary boolean functions with one or two arguments: AND ( $\wedge$ ), OR ( $\vee$ ), and NOT ( $\neg$ ). A boolean circuit over the *basis*  $B$  with  $m$  inputs and one output corresponds to a directed acyclic graph (DAG) with in-degree at most two and two types of nodes: *inputs* – nodes with in-degree 0 labelled with a binary variable  $x_i$  or with a constant 0 or 1, and *gates* – nodes with in-degree  $> 0$  labelled by one of the boolean functions  $\wedge$ ,  $\vee$ , and  $\neg$ . Further, the DAG has exactly one gate with out-degree 0 and this is designated the *output* node. For nodes  $i$  and  $j$ , an arc  $ij$  means that the value computed at  $i$  is used as an input to the gate or function at node  $j$ . The number of gates in a circuit  $C$  is its *size*, denoted as  $|C|$ . The only boolean circuits we discuss are defined over the basis  $B$ .

Each boolean circuit defines (or computes) a boolean function of its 0-1 inputs (or associated variables) in the following manner. Consider a topological ordering of the DAG representing the circuit, i.e., a linear ordering of its nodes in which each node precedes all nodes to which it has outgoing edges. Then process the gates according to this ordering: for each gate  $j$  place a value on  $j$  computed as the value of the boolean function at  $j$  applied to the values on nodes  $i$  with arcs  $ij$ . The 0-1 value on the output node (the last node in the ordering) is the value of the function computed by the circuit. An algorithm  $\mathcal{A}$  which has boolean inputs, a boolean output and whose elementary operations are the boolean operations in  $B$ , corresponds to a class  $\{C_n \mid n \geq 1\}$ , where  $C_n$  is a boolean circuit with  $n$  inputs. The running time of  $\mathcal{A}$  on inputs of size  $n$  equals  $|C_n|$ . If there is a polynomial  $p(n)$  such that  $|C_n| \leq p(n)$  for all  $n \geq 0$ , then  $\mathcal{A}$  is a polynomial-time algorithm.

A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is *monotone* if for  $x, y$  in  $\mathbb{R}^n$ ,  $x \leq y$  implies  $f(x) \leq f(y)$ . Here  $x \leq y$  stands for the conditions  $x_i \leq y_i$  for  $i = 1, \dots, n$ . *Monotone operations* are monotone functions with one or two inputs; some examples are

$$tx, \quad r + x, \quad x + y, \quad \lfloor x \rfloor, \quad thr(x, 0) \tag{1}$$

where  $t$  is a non-negative constant,  $x$  and  $y$  are real variables, and  $r$  is a real constant;  $thr(x, 0)$  is a *threshold function* which returns 0, if  $x < 0$ , and 1 otherwise. The boolean

functions  $\wedge$  and  $\vee$  are clearly monotone operations over the domain  $\{0, 1\}$  when their inputs and outputs are treated as numbers, whereas  $\neg$  is not a monotone operation. A *monotone boolean circuit* is similar to a boolean circuit, except that it uses only  $\wedge$  gates and  $\vee$  gates. A *monotone real circuit* is one with arbitrary monotone operations as gates.

Let  $CLIQUE_{k,n}$  (say  $k$  is a fixed function of  $n$ ) be the function which takes as input graphs on  $n$  nodes or vertices (given as incidence vectors of edges), and returns 1 if the input graph contains a clique of size  $k$ , and 0 if the graph contains a vertex-coloring of size  $k - 1$  (and is undefined for all other graphs). This function is monotone over the domain consisting of graphs containing a  $k$ -clique or a  $k - 1$  vertex-coloring as adding edges to a graph causes the maximum clique size to increase. Any monotone boolean circuit solving  $CLIQUE_{k,n}$ , when  $k$  is a certain function of  $n$ , has an exponential number of gates (see Razborov [23], and Alon and Boppana [2]). Pudlák, and independently Cook and Haken [6], proved a similar result for monotone real circuits.

**Theorem 1** [21] *Let  $C_n$  be a monotone real circuit computing  $CLIQUE_{k,n}$ , where  $k = \lfloor n^{2/3} \rfloor$ . Then  $|C_n| \geq 2^{\Omega((n/\log n)^{1/3})}$ .*

Pudlák [21] defined a system of linear inequalities  $\mathcal{I}$  (say in variables  $w$ ) such that if  $\mathcal{I}$  has a 0-1 solution, then there is a graph on  $n$  nodes which has both a clique of size  $k$  and a vertex-coloring of size  $k - 1$ . He proved that given a GC cutting plane proof  $\mathcal{P}$  of  $\mathbf{0}w \leq -1$  (which proves that  $\mathcal{I}$  has no 0-1 solution) of length  $L$ , one can construct a monotone real circuit with  $O(\text{poly}(L))$  gates solving  $CLIQUE_{k,n}$ . Theorem 1 then implies that  $L$  is exponential in  $n$ . Pudlák used the following properties of GC cuts in constructing monotone real circuits from GC cutting-plane proofs:

- (A) If  $gx + hy \leq d$  is a GC cut for  $Ax + By \leq c$ , then for any 0-1 vector  $y'$ ,  $gx \leq d - hy'$  is a GC cut for  $Ax \leq c - By'$ ;
- (B) if  $gx + hy \leq d$  is a GC cut for  $Ax \leq e$ ,  $By \leq f$ , then there are numbers  $r$  and  $s$  such that  $gx \leq r$  is a GC cut for  $Ax \leq e$ , and  $hy \leq s$  is a GC cut for  $By \leq f$ , and  $r + s \leq d$ ;
- (C) The number  $r$  (or  $s$ ) can be computed from  $A, e$  (or  $B, f$ ) with polynomially many monotone operations.

Property (A) is easy to prove. Consider property (B). As  $gx + hy \leq d$  is a GC cut for  $Ax \leq e$ ,  $By \leq f$ , we can assume that  $g, h, d$  are integral and there are multiplier vectors  $\lambda, \mu \geq 0$  such that  $g = \lambda A$ ,  $h = \mu B$ ,  $d = \lfloor \lambda e + \mu f \rfloor$ . Clearly,  $gx \leq \lfloor \lambda e \rfloor$  is a GC cut for  $Ax \leq e$ , and so is  $hy \leq \lfloor \mu f \rfloor$ , and  $\lfloor \lambda e \rfloor + \lfloor \mu f \rfloor \leq \lfloor \lambda e + \mu f \rfloor$ . Property (C) also follows from this; the number  $r$  in (C) equals  $\lfloor \lambda e \rfloor$  and if  $\lambda \in \mathbb{R}^m$ , then  $r$  can be computed from  $e$  via  $2m$  monotone operations from the list in (1) via the following algorithm:

1. For  $i = 1, \dots, m$  compute  $q_i := \lambda_i e_i$  (here the coefficients of  $\lambda$  are treated as non-negative constants).
2. Set  $q$  to  $q_1$  and for  $i = 2, \dots, m$  set  $q$  to  $q + q_i$ .
3. Finally, set  $r$  to  $\lfloor q \rfloor$ .

To construct a circuit computing  $\lfloor \lambda e \rfloor$  as a function of  $e$  from the above algorithm, just add inputs for  $e_1, \dots, e_m$  and a gate labelled with each of the  $2m$  monotone operations with an arc from gate  $i$  to gate  $j$  iff the operation at gate  $j$  uses the value computed by  $i$ .

Properties (A) and (B) hold for the matrix cuts of Lovász and Schrijver (cuts based on the  $N$  and  $N_+$  operators); see [22], [10]. Property (C) is often hard to prove, and is not known to hold for matrix cuts. In Lemma 5, we prove (via the equivalence of split cuts and MIR cuts) that slight variants of properties (B) and (C) hold for split cuts. We use Lemma 5 and ideas from Pudlák's paper to show in Theorem 8 that any split cut proof of  $\mathbf{0}w \leq -1$  from Pudlák's system  $\mathcal{I}$  has exponential length. For completeness, we give the inequality system  $\mathcal{I}$  below.

Let  $k = \lfloor n^{2/3} \rfloor$ . Let  $z$  be a vector of  $n(n-1)/2$  0-1 variables, such that every 0-1 assignment to  $z$  corresponds to the incidence vector of a graph on  $n$  nodes (assume nodes are numbered from  $1, \dots, n$ ). Let  $x$  be the 0-1 vector of variables ( $x_i : i = 1, \dots, n$ ) and let  $y$  be the 0-1 vector of variables ( $y_{ij} : i = 1, \dots, n, j = 1, \dots, k-1$ ). Let  $N = \{1, \dots, n\}$ . Consider the inequalities

$$\sum_i x_i \geq k, \tag{2}$$

$$x_i + x_j \leq 1 + z_{ij}, \quad \forall i, j \in N, \text{ with } i < j, \tag{3}$$

$$\sum_{j=1}^{k-1} y_{ij} = 1, \quad \forall i \in N, \tag{4}$$

$$y_{is} + y_{js} \leq 2 - z_{ij}, \quad \forall i, j \in N \text{ with } i < j, \text{ and } \forall s \in \{1, \dots, k-1\}. \tag{5}$$

In any 0-1 solution of the above inequalities, the set of nodes  $\{i : x_i = 1\}$  forms a clique of size  $k$  or more, and for all  $j \in \{1, \dots, k-1\}$ , the set  $\{i : y_{ij} = 1\}$  is a stable set. Thus, the variables  $y_{ij}$  define a mapping of nodes in a graph to  $k-1$  colors in a proper vertex-colouring. Therefore the inequalities (2) - (5) have no 0-1 solution. Let  $Ax + Cz \leq e$  stand for the inequalities (2) and (3), along with the bounds  $0 \leq x \leq 1$ . Let  $By + Dz \leq f$  stand for the inequalities (4) and (5), along with the bounds  $0 \leq y \leq 1$  and  $0 \leq z \leq 1$ . Note that the above inequalities have  $O(n^2)$  variables and  $O(n^3)$  constraints; for technical reasons we need the fact that  $C \leq 0$ . Theorem 1 implies that every monotone real circuit which takes graphs on  $n$  nodes as input (in the form of a 0-1 vector  $z'$ ) and outputs 0 or 1 such that output 0 implies  $Ax \leq e - Cz'$  has no 0-1 solution and output 1 implies  $By \leq f - Dz'$  has no 0-1 solution, has exponentially many gates.

### 3 Mixed-integer rounding cuts

For a number  $v$  and an integer  $t$ , define  $\hat{v} = v - \lfloor v \rfloor$ . Define

$$Q = \{v \in \mathbb{R}, z \in \mathbb{Z} : v + z \geq b, v \geq 0\}.$$

Wolsey [25] defined the *basic mixed-integer inequality* as

$$v + \hat{b}z \geq \hat{b}\lceil b \rceil, \tag{6}$$

and showed that it is valid and facet-defining for  $Q$ ; also see [19]. In Figure 1(a), we depict the points in  $Q$  (when  $b$  is not integral) by horizontal lines. In Figure 1(b), the half-plane above the dashed line represents (6), and contains the shaded regions which are  $Q \cap \{z \leq \lfloor b \rfloor\}$  and  $Q \cap \{z \geq \lceil b \rceil\}$ . Therefore (6) is a split cut for  $Q$ . The following

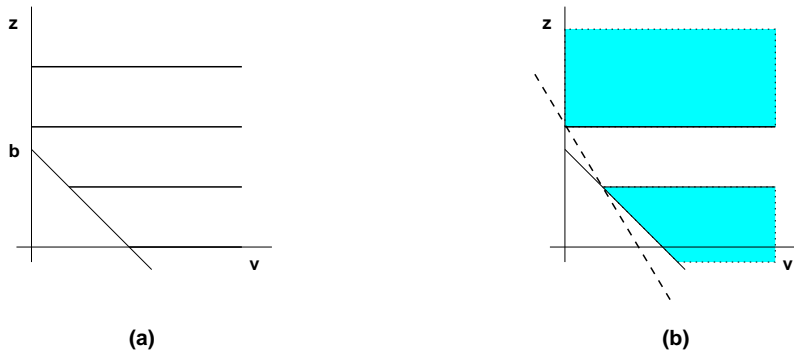


Figure 1: The basic mixed-integer inequality

is a well-known property of the (6) and can easily be inferred from Figure 1.

**Lemma 2**  $\text{conv}(Q) = \{v, z \in \mathbb{R} : v + z \geq b, v + \hat{b}z \geq \hat{b}\lceil b \rceil, v \geq 0\}$ .

Therefore, any linear inequality satisfied by points in  $Q$  is implied by a non-negative linear combination of the inequalities  $v \geq 0$ ,  $v + z \geq b$  and (6).

Let  $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ , and assume  $A$  has  $m$  rows. Define  $s = b - Ax$ ; clearly  $s \geq 0$  for all  $x \in P$ . Let  $\lambda \in \mathbb{R}^m$  be a row vector such that  $\lambda A$  is integral. Define  $\beta$  as  $\lambda b$ , and  $\lambda^+$  by  $\lambda_i^+ = \max\{\lambda_i, 0\}$ . Then the equation  $\lambda s + (\lambda A)x = \lambda b$  is valid for  $P$  and so is the inequality

$$\lambda^+ s + (\lambda A)x \geq \lambda b \tag{7}$$

For all points in  $P$ ,  $\lambda^+ s$  is non-negative and  $(\lambda A)x$  is integral. The basic mixed-integer inequality (6) implies that

$$\lambda^+ s + \hat{\beta}(\lambda A)x \geq \hat{\beta}(\lceil \beta \rceil), \tag{8}$$

or equivalently  $\lambda^+(b - Ax) + \hat{\beta}(\lambda A)x \geq \hat{\beta}(\lceil \beta \rceil)$ , is a valid inequality for integral points in  $P$  (recall that  $\hat{\beta} = \beta - \lfloor \beta \rfloor$ ). This inequality is the *mixed-integer rounding (MIR)* cut for  $P$ . It follows from the discussion above that the MIR cut is a split cut for  $P$  derived from the disjunction  $(\lambda A)x \leq \lfloor \beta \rfloor \vee (\lambda A)x \geq \lceil \beta \rceil$ . Nemhauser and Wolsey defined [20] MIR cuts and proved that every split cut for  $P$  is also an MIR cut for  $P$ ; see [12] for different ways of defining the MIR cut, and the equivalence of (8) with the definition in [20].

**Definition 3** *The split closure of a polyhedron  $P$ , denoted by  $sc(P)$ , is the set of points in  $P$  satisfying all split cuts for  $P$ .*

The MIR closure (denoted by  $P^{MIR}$ ) is defined similarly in terms of MIR cuts. The split closure of a polyhedron is therefore the same as its MIR closure.

## 4 Complexity of split-cut proofs

We start off by proving the properties (A) and (B) given in Section 2 for split cuts. To do so, we invoke the equivalence of split cuts and MIR cuts, and use the lemma below. For  $i = 1, 2$ , let

$$P_i = \{v_i, z_i : v_i + z_i \geq b_i, v_i \geq 0, z_i \in \mathbb{Z}\}, \quad (9)$$

**Lemma 4** *Let  $P_1, P_2$  be defined as above. Let  $b_3 = b_1 + b_2$ . Then the inequality  $(v_1 + v_2) + \hat{b}_3(z_1 + z_2) \geq \hat{b}_3 \lceil b_3 \rceil$  valid for  $P_1 \cap P_2$  is implied by a non-negative combination of the inequalities defining  $P_1$  and  $P_2$  and the basic mixed-integer inequalities for  $P_1$  and  $P_2$ .*

**Proof** By Lemma 2, for  $i = 1, 2$ ,

$$\text{conv}(P_i) = \{v_i, z_i : v_i + z_i \geq b_i, v_i + \hat{b}_i z_i \geq \hat{b}_i \lceil b_i \rceil, v_i \geq 0\}.$$

Observe that all points  $(v_1, v_2, z_1, z_2) \in P_1 \cap P_2$  satisfy  $v_1 + v_2 + z_1 + z_2 \geq b_1 + b_2$  and  $v_1 + v_2 \geq 0$  and  $z_1 + z_2 \in \mathbb{Z}$ . Using (6), we see that

$$v_1 + v_2 + \hat{b}_3(z_1 + z_2) \geq \hat{b}_3 \lceil b_3 \rceil. \quad (10)$$

is valid for  $P_1 \cap P_2$ . Therefore

$$\begin{aligned} \hat{b}_3 \lceil b_3 \rceil &\leq \min\{v_1 + v_2 + \hat{b}_3(z_1 + z_2) : (v_1, v_2, z_1, z_2) \in P_1 \cap P_2\} \\ &\leq \min\{v_1 + \hat{b}_3 z_1 : (v_1, z_1) \in P_1\} + \min\{v_2 + \hat{b}_3 z_2 : (v_2, z_2) \in P_2\}, \end{aligned}$$

as optimal solutions  $(v_1^*, z_1^*)$  and  $(v_2^*, z_2^*)$  of the last two minimization problems with optimal values  $\mu_1$  and  $\mu_2$ , yield a feasible solution  $(v_1^*, v_2^*, z_1^*, z_2^*)$  of the first minimization problem with objective value  $\mu_1 + \mu_2$ . As

$$v_1 + \hat{b}_3 z_1 \geq \mu_1 \text{ and } v_2 + \hat{b}_3 z_2 \geq \mu_2$$

are valid inequalities for  $\text{conv}(P_1)$  and  $\text{conv}(P_2)$ , respectively, (10) is implied by a non-negative combination of inequalities defining  $\text{conv}(P_1)$  and  $\text{conv}(P_2)$ .  $\blacksquare$

**Lemma 5** *Let  $x, y$  be vectors of integer variables, with no common components. Let*

$$Q_1 = \{x : Ax \leq e\}, \quad Q_2 = \{y : Cy \leq f\},$$

$$Q_3 = \{(x, y) : x \in Q_1, y \in Q_2\} = Q_1 \cap Q_2,$$

where  $A, C, e, f$  are matrices with appropriate dimensions. Let  $ax + cy \leq d$  be a split cut for  $Q_3$ . Let  $g = \max\{ax : x \in \text{sc}(Q_1)\}$  and  $h = \max\{cy : y \in \text{sc}(Q_2)\}$ . Then  $g + h \leq d$ .

**Proof** Let  $ax + cy \leq d$  be a split cut for  $Q_3$ . It is also an MIR cut derived from the system

$$Ax + s = e, \quad Cy + t = f, \quad s, t \geq 0.$$

More precisely, there are vectors  $\lambda$  and  $\mu$  such that the split cut above equals

$$\lambda^+ s + \mu^+ t + \hat{\beta}((\lambda A)x + (\mu C)y) \geq \hat{\beta} \lceil \beta \rceil, \quad (11)$$

$$\text{where } \beta = \lambda e + \mu f, \quad \lambda A \text{ and } \mu C \text{ are integral.} \quad (12)$$

Define

$$\begin{aligned} v_1 &= \lambda^+ s, & z_1 &= \lambda Ax, & b_1 &= \lambda e, \\ v_2 &= \mu^+ t, & z_2 &= \mu Cy, & b_2 &= \mu f. \end{aligned}$$

Clearly  $\beta = b_1 + b_2$ . Defining  $P_1, P_2$  in Lemma 4 in terms of the variables  $v_1, v_2, z_1, z_2$ , and letting  $b_3 = \beta$ , we see that (11) can be written as

$$v_1 + v_2 + \hat{b}_3(z_1 + z_2) \geq \hat{b}_3 \lceil b_3 \rceil.$$

By the proof of Lemma 4, there are numbers  $\mu_1$  and  $\mu_2$  such that  $\mu_1 + \mu_2 \geq \hat{b}_3 \lceil b_3 \rceil$  and for  $i = 1, 2$ ,  $v_i + \hat{b}_3 z_i \geq \mu_i$  is a non-negative combination of the inequalities defining  $P_i$  and the basic mixed-integer inequality for  $P_i$ . Consider the case  $i = 1$ . Substituting out the slacks  $s$  and  $t$ , we see that  $v_1 \geq 0$  and  $v_1 + z_1 \geq b_1$  are both implied by non-negative combinations of the inequalities defining  $Q_1$ , and  $v_1 + \hat{b}_1 z_1 \geq \hat{b}_1 \lceil b_1 \rceil$  defines an MIR cut for  $Q_1$ . Finally,  $v_1 + \hat{b}_3 z_1 \geq \mu_1$  becomes  $ax \leq g'$  for some  $g'$ .

Therefore, substituting out the slacks  $s, t$ , we conclude that there are numbers  $g'$  and  $h'$  such that

- (i)  $g' + h' \leq d$ ,
- (ii)  $ax \leq g'$  is a non-negative linear combination of the inequalities defining  $Q_1$  and some MIR cut for  $Q_1$ ,

(iii)  $cx \leq h'$  is a non-negative linear combination of the inequalities defining  $Q_2$  and some MIR cut for  $Q_2$ .

If we define

$$g = \max\{ax : x \in Q_1^{MIR}\}, \quad h = \max\{cy : y \in Q_2^{MIR}\},$$

$$\text{then } g \leq g', \quad h \leq h' \text{ and } g + h \leq d.$$

But recall that  $sc(Q_1) = Q_1^{MIR}$  and  $sc(Q_2) = Q_2^{MIR}$ . The lemma follows.  $\blacksquare$

It may be possible to obtain a direct proof of Lemma 5 without using the equivalence of split cuts and MIR cuts, but we believe it is not trivial.

**Definition 6** *A split cut proof of  $cx \leq d$  from  $Ax \leq b$  is a simplified split cut proof if every inequality in the proof is either a non-negative linear combination of previous inequalities in the proof and  $Ax \leq b$  or a split cut derived from two inequalities from among the previous inequalities and  $Ax \leq b$ .*

**Lemma 7** *Given a split cut proof of length  $L$  of some inequality from  $\bar{A}x \leq \bar{b}$ , there is a simplified split cut proof of the same inequality with length  $3L$ .*

**Proof** Let  $cx \leq d$  be a split cut for  $P = \{x : Ax \leq b\}$ , and assume  $A$  has  $m$  rows. Then  $cx \leq d$  is valid for  $P_1 = P \cap \{\alpha x \leq \beta\}$  and  $P_2 = P \cap \{\alpha x \geq \beta + 1\}$ , for some integral row vector  $\alpha$  and integer  $\beta$ . There are multipliers  $\lambda_1, \lambda_2 \in \mathbb{R}^m$  and  $\mu_1, \mu_2 \in \mathbb{R}$  with  $\lambda_1, \lambda_2, \mu_1, \mu_2 \geq 0$  such that

$$\begin{aligned} c &= \lambda_1 Ax + \mu_1 \alpha, & c &= \lambda_2 Ax - \mu_2 \alpha, \\ d &\geq \lambda_1 b + \mu_1 \beta, & d &\geq \lambda_2 b - \mu_2(\beta + 1). \end{aligned}$$

Clearly  $cx \leq d$  is a split cut for  $\{x : \lambda_1 Ax \leq \lambda_1 b, \lambda_2 Ax \leq \lambda_2 b\}$ . A simplified split cut proof can be obtained by replacing each split cut in the proof with three inequalities: the inequalities  $\lambda_1 Ax \leq \lambda_1 b$  and  $\lambda_2 Ax \leq \lambda_2 b$ , followed by the split cut itself.  $\blacksquare$

If  $cx \leq d$  is a split cut for  $P$  as in the proof above, we say it is derived from the split disjunction  $\alpha x \leq \beta \vee \alpha x \geq \beta + 1$  using the multipliers  $\lambda_1, \lambda_2$  and  $\mu_1, \mu_2$ .

We now have all the tools required to show that any split cut proof of the 0-1 infeasibility of the system  $\mathcal{I}$  described in Section 2 has exponential length. The next theorem is a straightforward application of Pudlák's proof technique for GC cutting plane proofs.

**Theorem 8** *Assume there exists a simplified split cut proof of  $\mathbf{0}x + \mathbf{0}y + \mathbf{0}z \leq -1$  from  $\mathcal{I} = \{x, y, z : Ax + Cz \leq e, By + Dz \leq f\}$  of length  $L$ . Then there exists a monotone real circuit of size  $O(Ln^2)$  solving  $CLIQUE_{k,n}$ .*

**Proof** Let  $m$  stand for the number of inequalities defining the set  $\mathcal{I}$ . Let  $\mathcal{R} = (\mathcal{R}_1, \dots, \mathcal{R}_{m+L})$  be a sequence of  $m + L$  inequalities such that  $\mathcal{R}_1, \dots, \mathcal{R}_m$  are the inequalities in  $Ax + Cz \leq e$  and  $By + Dz \leq f$ , and  $\mathcal{R}_{m+1}, \dots, \mathcal{R}_{m+L}$  are the inequalities in the simplified split cut proof assumed to exist in the theorem. Denote  $\mathcal{R}_i$ , the  $i$ th inequality in  $\mathcal{R}$ , by  $a_i x + b_i y + c_i z \leq d_i$ . Then  $\mathcal{R}_{m+L}$  equals  $\mathbf{0}x + \mathbf{0}y + \mathbf{0}z \leq -1$ . Let  $I_i$  stand for  $\{1, \dots, i - 1\}$ . By definition, for  $i > m$   $\mathcal{R}_i$  is either a split cut derived from  $\mathcal{R}_k$  and  $\mathcal{R}_l$  for some  $k, l \leq i - 1$ , or a non-negative linear combination of the inequalities  $\mathcal{R}_1, \dots, \mathcal{R}_{i-1}$  with the multipliers  $\lambda_{ij} \geq 0$  ( $j \in I_i$ ). In the latter case, we can assume, by Caratheodory's theorem, that at most  $O(n^2)$   $\lambda_{ij}$  are non-zero.

Let  $z'$  stand for a 0-1 assignment to  $z$ . The inequality sequence  $\mathcal{R}'$ , where

$$\mathcal{R}'_i \text{ is } a_i x + b_i y \leq d_i - c_i z',$$

is a simplified split cut proof of  $\mathbf{0}x + \mathbf{0}y \leq -1$  from  $Ax \leq e - Cz'$  and  $By \leq f - Dz'$  with the same length as  $\mathcal{R}$ :  $\mathcal{R}'_i$  can be derived with the same multipliers as  $\mathcal{R}_i$ . Define  $d'_i = d_i - c_i z'$ . We construct a sequence of inequalities  $\mathcal{S}$  involving only  $x$ , and another sequence  $\mathcal{T}$ , involving only  $y$ , such that

$$\mathcal{S}_i \equiv a_i x \leq g_i, \quad \mathcal{T}_i \equiv b_i y \leq h_i, \quad \text{and } g_i + h_i \leq d'_i;$$

$$\mathcal{S}_i, \mathcal{T}_i \text{ are valid for integral solutions of } Ax \leq e - Cz', By \leq f - Dz'. \quad (13)$$

Thus  $\mathcal{S}_i + \mathcal{T}_i$  has the same left hand side as  $\mathcal{R}'_i$ , but an equal or smaller right-hand side.

For  $i = 1, \dots, m$ , if  $\mathcal{R}'_i$  involves only  $x$ , then set  $\mathcal{S}_i$  to  $\mathcal{R}'_i$  and  $\mathcal{T}_i$  to  $\mathbf{0}y \leq 0$ , otherwise set  $\mathcal{S}_i$  to  $\mathbf{0}x \leq 0$  and  $\mathcal{T}_i$  to  $\mathcal{R}'_i$ . Define subsequent terms of  $\mathcal{S}$  and  $\mathcal{T}$  as follows. For  $i = m + 1, \dots, m + L$ , if  $\mathcal{R}'_i$  is a non-negative linear combination of inequalities  $\mathcal{R}'_j$  ( $j \in I_i$ ) with the multipliers  $\lambda_{ij} > 0$  ( $j \in I_i$ ), then let  $\mathcal{S}_i$  and  $\mathcal{T}_i$  be non-negative linear combinations of  $\mathcal{S}_j$  ( $j \in I_i$ ) and  $\mathcal{T}_j$  ( $j \in I_i$ ), respectively, with the same multipliers  $\lambda_{ij}$  ( $j \in I_i$ ). If  $\mathcal{R}'_i$  is a split cut derived from  $\mathcal{R}'_k$  and  $\mathcal{R}'_l$  for some  $k, l \leq i - 1$ , then define

$$Q_1 = \{x : a_k x \leq g_k, a_l x \leq g_l\}, \quad Q_2 = \{y : c_k y \leq h_k, c_l y \leq h_l\}.$$

It follows from Lemma 5 that  $\mathcal{R}'_i$  is implied by the inequalities defining  $Q_1$  and  $Q_2$  and some split cuts for these sets. More precisely, if we define

$$g_i = \max\{a_i x : x \in sc(Q_1)\}, \quad h_i = \max\{c_i y : y \in sc(Q_2)\}, \quad (14)$$

then  $g_i + h_i \leq d'_i$ . We then define  $\mathcal{S}_i$  to be  $a_i x \leq g_i$ , and  $\mathcal{T}_i$  to be  $c_i y \leq h_i$ .

Observe that the inequality  $g_i + h_i \leq d'_i$  in (13) is by definition true for  $i = 1, \dots, m$ ; either  $g_i = d'_i$  and  $h_i = 0$ , or  $h_i = d'_i$  and  $g_i = 0$ . Let  $i > m$ , and assume by induction that (13) is true for smaller values of  $i$ . If  $\mathcal{R}'_i$  is a non-negative combination of inequalities, then (13) is clearly true. If  $\mathcal{R}'_i$  is a split cut derived from two previous inequalities, then (13) holds by (14) and Lemma 5. Therefore the last inequalities in  $\mathcal{S}$  and  $\mathcal{T}$  are, respectively,  $\mathbf{0}x \leq g_{m+L}$  and  $\mathbf{0}y \leq h_{m+L}$ . As  $d'_{m+L} = d_{m+L} = -1$ , one of  $g_{m+L}$  and

$h_{m+L}$  is less than 0, and we have a proof of integer infeasibility of either  $Ax \leq e - Cz'$  or  $By \leq f - Dz'$ .

We use  $\mathcal{R}$  and its coefficients to define a monotone circuit  $\mathcal{C}$  which takes as input a 0-1 vector  $z'$  and outputs  $\text{thr}(g_{m+L}, 0)$ .  $\mathcal{C}$  has inputs labelled by  $z_{ij}$  for all  $i, j \in N = \{1, \dots, n\}$ . Its gates are labelled by the monotone operations in (1) and the functions

$$f_{a_k, a_l, a_i}(\delta, \gamma) = \max\{a_i x : x \in \text{sc}(\{x : a_k x \leq \delta, a_l x \leq \gamma\})\}$$

where  $a_k, a_l, a_i$  are the coefficients of the  $x$  variables in  $\mathcal{R}_k, \mathcal{R}_l$  and  $\mathcal{R}_i$ , respectively. Note that the functions  $f_{a_k, a_l, a_i}(\delta, \gamma)$  are monotone operations of  $\delta$  and  $\gamma$ : if  $\delta \leq \delta'$  and  $\gamma \leq \gamma'$ , then

$$P = \{x : a_k x \leq \delta, a_l x \leq \gamma\} \subseteq P' = \{x : a_k x \leq \delta', a_l x \leq \gamma'\}$$

and therefore  $\text{sc}(P) \subseteq \text{sc}(P')$ .

$\mathcal{C}$  first computes  $g_1, \dots, g_m$ ; by definition  $g_k$  with  $1 \leq k \leq m$  is of the form  $e_i - (Cz)_i$  when  $\mathcal{R}_k$  is the  $i$ th inequality in  $Ax + Cz \leq e$ , and is 0 otherwise. Again by the definition of  $Ax + Cz \leq e$ , any non-constant component of  $e - Cz$  simply has the form  $1 + z_{ij}$  (see equation (3)), and therefore each  $g_k$  with  $1 \leq k \leq m$  can be computed with a single monotone operation in the list in (1) from the inputs  $z_{ij}$ .

For  $i > m$ , if

$$\mathcal{R}_i = \sum_{j \in I_i: \lambda_{ij} > 0} \lambda_{ij} \mathcal{R}_j, \text{ then } g_i = \sum_{j \in I_i: \lambda_{ij} > 0} \lambda_{ij} g_j.$$

Therefore we can compute  $g_i$  from  $g_j (j \in I_i)$  using at most  $2O(n^2) = O(n^2)$  monotone operations as in the computation of  $\lfloor \lambda e \rfloor$  in Section 2 (the  $\lambda_{ij}$ s are fixed as  $\mathcal{R}$  is fixed). If  $\mathcal{R}_i$  is a split cut derived from  $\mathcal{R}_k, \mathcal{R}_l$  with  $k, l \in I_i$ , then  $\mathcal{C}$  computes  $g_i$  as  $g_i = f_{a_k, a_l, a_i}(g_k, g_l)$ , as in (14). Finally, the circuit outputs  $\text{thr}(g_{m+L}, 0)$ , which is a monotone operation. The computation of each  $g_i$  requires at most  $O(n^2)$  monotone operations, and thus  $\mathcal{C}$  can be assumed to have  $O(Ln^2)$  monotone gates. Note that  $\mathcal{C}$  does not compute  $h_1, \dots, h_{m+L}$ ; this is because  $f - Dz'$  is not a monotone function of  $z'$ . By definition, if the circuit returns 0 for some input  $z'$ , then  $g_{m+L} < 0$  and  $Ax \leq e - Cz'$  has no integral solutions. If the output is 1, then  $h_{m+L}$  must be negative (if it were computed) and  $By \leq f - Dz'$  has no integral solutions.  $\blacksquare$

**Corollary 9** *Every split cut proof of  $0x + 0y + 0z \leq -1$  from  $\mathcal{I}$  has exponential length.*

A natural question is: why do we prove the above result for split cuts, and not directly for MIR cuts, given that we use MIR cuts to prove Lemma 5? One can avoid introducing split cuts, and work entirely with MIR cuts, but one has to carefully deal with the additional slack variables in the definition of the MIR cut (8) to ensure that only monotone operations are used to generate  $\mathcal{S}$ .

It is shown in [10, Lemma 5.7] that a branch-and-cut proof of (integer) infeasibility  $\mathcal{R}$  using lift-and-project cuts and GC cuts and branching on 0-1 variables can be transformed into a cutting plane proof of infeasibility  $\mathcal{S}$  with length  $s + t$ , where  $s$  and  $t$  are the number of cuts and branching decisions in  $\mathcal{R}$ , respectively. In this proof, every branching decision is replaced by a lift-and-project cut. One can use the proof technique for the result above and show that if there exists a branch-and-cut proof of the fact that a polyhedron  $P$  has no integral solutions using  $s$  split cuts, and branching on 0-1 variables with  $t$  branching decisions, there exists a split-cut proof of the same fact with length  $s + t$ . Therefore, any branch-and-cut proof of integer infeasibility of  $\mathcal{I}$  has exponentially many cuts plus nodes.

Many cutting planes discussed in the literature – besides GC cuts and lift-and-project cuts – are either special cases of split cuts (see [19]) or can be derived via a short (polynomial in the number of variables) split cut proof. For example, the pairing inequalities [15] form a subclass of split cuts. The two-step MIR inequalities in [11] can be obtained by a split cut proof of length 2. Any cutting plane proof of the integer infeasibility of  $\mathcal{I}$  using the above cuts has exponential length.

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