Asymptotic Analysis of a Contraction Mapping Algorithm for Multiple Frequency Estimation[1]

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Abstract

Based on an asymptotic analysis of the contraction mapping (CM) method of Li and Kedem (*IEEE Trans. Inform. Theory*, vol. 39, pp. 989–998, 1993), a bandwidth shrinkage rule is proposed for fast and accurate estimation of the frequencies of multiple sinusoids from noisy measurements. The CM frequency estimates are defined as the fixed-points of a contractive mapping formed by the lag-one autocorrelation coefficient calculated from the output of a parametric filter applied to the observed time series. With judiciously chosen bandwidth parameters according to the asymptotic analysis, the algorithm is shown to be able to accommodate possibly poor initial values of precision $\mathcal{O}(n^{-1/3})$ and converge to a final estimate whose accuracy is arbitrarily close to $\mathcal{O}(n^{-3/2})$, the optimal error rate for frequency estimation under the Gaussian assumption. The total computational complexity of the algorithm is shown to be $\mathcal{O}(n\log n)$, which is comparable to that of n-point FFT. A novelty in the asymptotic analysis is that it accommodates closely-spaced frequencies by allowing not only the filter bandwidth but also the frequency separation to be functions of the sample size n. This enables an assessment of the accuracy of the frequency estimates for given bandwidths and initial values in situations where some or all of the frequencies are close to each other.

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Running Head. Multiple Frequency Estimation

1 Introduction

Consider a time series $\{y_1, \cdots, y_n\}$ obtained from the following random process:

$$y_t := \sum_{k=1}^{p} \beta_k \cos(\omega_k t + \phi_k) + \varepsilon_t, \tag{1.1}$$

where β_k, ω_k, and ϕ_k are unknown constants satisfying $\beta_k > 0$, $0 < \omega_1 < \cdots < \omega_p < \pi$, and $\phi_k \in (-\pi, \pi]$, and $\{\varepsilon_t\}$ is a zero-mean stationary process. This 'multiple-sinusoid-plus-noise' model has important scientific and engineering applications in, for example, radar and sonar signal processing and rotating machinery.

A fundamental problem in these applications is to accurately estimate the unknown frequencies ω_k. In particular, an accuracy of $\mathcal{o}_P(n^{-1})$ is required for reliable assessment of the amplitudes of the sinusoids, as demonstrated in [1] and [2]. Traditional methods of obtaining such accurate frequency estimates include the maximization of periodogram (MP) as a continuous function of the frequency variable and the minimization of the error sum of squares by nonlinear least-squares (NLS) regression (which coincides with the maximum likelihood method if $\{\varepsilon_t\}$ is Gaussian white noise). Both MP and NLS are statistically efficient for frequency estimation in the sense that the estimation error achieves asymptotically the Cramér-Rao lower bound (derived under the Gaussian white noise assumption) that can be expressed as $\mathcal{O}_P(n^{-3/2})$ (e.g., [3]–[7]). Unfortunately, the computational requirements of these methods are quite prohibitive, not only because iterative optimization algorithms are needed to compute the estimates, but more importantly because extremely precise initial values, typically of accuracy $\mathcal{o}(n^{-1})$, which cannot be obtained by n-point FFT, are required to ensure convergence (e.g., [1], [4], [8], and [9]). Furthermore, the MP and NLS estimates cannot be easily updated upon the arrival of new observations without re-processing the entire data record. These considerations have motivated the proposal of many alternative methods in both statistical and signal processing literature.

Iterative filtering (IF) is a favorite approach in signal processing to developing alternative methods of frequency estimation that are computationally efficient (e.g., [10]–[15]). A typical IF algorithm repeats the steps of enhancing the sinusoids with a bandpass filter and estimating the frequencies on the basis of the filtered data. Since recursive filters are often employed by IF algorithms, the frequency estimates can be easily updated upon the arrival of new observations in order to track possible frequency changes (e.g., [16]–[19]). The general premise of the IF approach is that as the frequency estimates become more accurate, the filter, which depends on the frequency estimates, would enhance the sinusoids more effectively and thus further improve the precision of frequency estimation in the subsequent cycle of iteration. This, indeed, has been vindicated by many numerical studies in the literature. What remains largely an open question is how to design the filter on the basis of the available frequency estimates so that the entire iterative scheme would converge to a solution of improved accuracy.

Because the sinusoids are localized in the frequency domain, bandpass filters are often employed to enhance them. One can use a filter with multiple passbands to simultaneously estimate all frequencies (e.g., [14] and [20]), or a filter with single passband to sequentially estimate each frequency (e.g., [21]). The first approach may have higher frequency resolution, as indicated by many numerical studies, but at the expense of greater computational complexity. In this paper, we focus on the second approach.

In essence, the second approach is an application of single-frequency estimation methods to the multiple frequency case by regarding all but one sinusoids as interference and lumping them into the noise term in (1.1). Since this approach relies on the bandpass filter to suppress both the noise and the interfering sinusoids, the bandwidth selection becomes an important issue. If the bandwidth is too large, the noise and the interfering sinusoids would not be effectively suppressed and the resulting frequency estimates would be inaccurate. On the other hand, if the bandwidth is too small, the desired sinusoid could be filtered out by a filter designed on the basis of poor frequency estimates and the iteration would not converge to the desired solution.

The main contribution of this paper is to analytically quantify the role of bandwidth in determining the required initial precision that ensures the convergence of an IF algorithm and the accuracy of the final frequency estimates after convergence. The IF algorithm that we focus on in this paper is the contraction mapping (CM) method of Li and Kedem [22]. This method employs a second-order autoregressive (AR) filter endowed with a bandwidth parameter (for other filters, see [23]–[25]). Statistical and numerical properties of the CM method in the case of single sinusoid have been studied by Li and Kedem [22], Li, Kedem, and Yakowitz [26], and most recently, by Song and Li [2], [27]. These studies show that if the bandwidth is judiciously adjusted with the iteration, the CM method can accommodate poor initial guesses of accuracy $\mathcal{O}_P(1)$ and converge to a final frequency estimate whose accuracy is arbitrarily close to $\mathcal{O}_P(n^{-3/2})$.

To investigate the CM method in the case of multiple sinusoids, one has to overcome two major obstacles. First, the interfering sinusoids have very different statistical properties from the noise (e.g., discrete versus continuous spectrum). Second, the interfering frequencies may reside in a close vicinity of the frequency to be estimated. To deal with the first problem, the interaction of the sinusoids among themselves and with the noise has to be carefully evaluated. To accommodate the second problem, we assume that the minimum distance among the frequencies may depend on the sample size n and may decrease to zero as n tends to infinity. Under this assumption, the bandwidth must also depend on the separation of the frequencies in order to suppress the interference. Consequently, the required initial precision for the CM iteration to converge depends not only on the bandwidth parameter but also on the frequency separation. It is shown that when the frequencies are not too close to each other (as compared to the filter bandwidth), the CM method retains its capability of producing accurate frequency estimates whose accuracy is arbitrarily close to $\mathcal{O}_P(n^{-3/2})$. The convergence is guaranteed as long as the initial precision is $\mathcal{o}_P(n^{-1/3})$. This requirement is easily satisfied by any root-n consistent estimates, including those produced by the multivariate IF method in [14] and by the singular-value-decomposition-based methods such as MUSIC and ESPRIT (e.g., [28]).

The rest of the paper is organized as follows. In Sec. 2, we introduce the CM frequency estimator. In Sec. 3, we present our main contributions in the form of five theorems and a resulting bandwidth shrinkage rule that leads to a three-step algorithm capable of improving poor initial values of accuracy $\mathcal{o}_P(n^{-1/3})$ to produce a final frequency estimator whose accuracy is arbitrarily close to $\mathcal{O}_P(n^{-3/2})$. A simulation example is also given in this section to demonstrate the algorithm. The remaining sections are devoted entirely to the mathematical proofs of the main results and are organized progressively in terms of technical complexity. More specifically, Sec. 4 contains the proof of the theorems on the basis of some preliminary propositions. The propositions are then proved in Sec. 5. Finally, some technical lemmas needed to prove the propositions are given and proved in Sec. 6.

2 The CM Frequency Estimator

For any given $\eta \in (0,1)$ and $\alpha := \cos\omega \in \mathcal{A} := (-2\eta(1+\eta^2)^{-1}, 2\eta(1+\eta^2)^{-1})$, let $\{y_t(\alpha)\}$ be obtained recursively from the observations $\{y_1, \cdots, y_n\}$ according to

$$y_t(\alpha) + 2\,\theta(\alpha)\,\eta\, y_{t-1}(\alpha) + \eta^2 y_{t-2}(\alpha) = y_t \quad (t = 1, \cdots, n), \tag{2.2}$$

where $y_{-1}(\alpha) = y_0(\alpha) := 0$ and

$$\theta(\alpha) := -\frac{1+\eta^2}{2\eta}\,\alpha := -\cos\lambda. \tag{2.3}$$

Note that (2.2) defines a causal stable AR(2) filter with transfer function $(1 + 2\,\theta(\alpha)\,\eta\,\mathfrak{B} + \eta^2\mathfrak{B}^2)^{-1}$, where $\mathfrak{B}$ is the backward-shift operator such that $\mathfrak{B}y_t = y_{t-1}$. Note also that $\lambda \in (0,\pi)$ in (2.3) is uniquely determined by $\eta \in (0,1)$ and $\alpha \in \mathcal{A}$.

Let the lag-one autocorrelation coefficient of $\{y_t(\alpha)\}$ be estimated by

$$\rho_n(\alpha) := \frac{\sum_{t=1}^n y_{t-1}(\alpha)\,\{y_t(\alpha) + \eta^2 y_{t-2}(\alpha)\}}{(1+\eta^2)\sum_{t=1}^n y_{t-1}^2(\alpha)}. \tag{2.4}$$

This estimator minimizes the weighted sum of the forward and backward prediction error sums of squares defined by $e_n^2(\rho) := \sum_{t=1}^n \{y_t(\alpha) - \rho\, y_{t-1}(\alpha)\}^2 + \eta^2 \sum_{t=1}^n \{y_{t-2}(\alpha) - \rho\, y_{t-1}(\alpha)\}^2$, where η^2 plays the role of a weight that discounts the contribution of the backward prediction errors. The CM method in [22] produces the frequency estimates from the fixed-point iteration

$$\hat{\alpha}_n^{(m)} := \rho_n(\hat{\alpha}_n^{(m-1)}) \quad (m = 1, 2, \cdots). \tag{2.5}$$

Suppose that with an initial guess $\hat{\alpha}_n^{(0)}$ in some neighborhood of $\alpha_k := \cos\omega_k$ the sequence $\{\hat{\alpha}_n^{(m)}\}$ converges to a fixed-point $\hat{\alpha}_n$ as $m \to \infty$. Then, since $\hat{\alpha}_n$ can be regarded as an estimator of α_k, the frequency $\omega_k = \arccos(\alpha_k)$ can be estimated by

$$\hat{\omega}_n := \arccos(\hat{\alpha}_n). \tag{2.6}$$

The convergence of (2.5) depends crucially on how close the initial value $\hat{\alpha}_n^{(0)}$ is to α_k. In other words, it depends on the accuracy of $\hat{\alpha}_n^{(0)}$ as an estimator of α_k. This initial accuracy required for convergence is in turn determined by the bandwidth parameter η. Numerical experiments in [14] indicate that the closer is η to unity, the more stringent is the requirement on $\hat{\alpha}_n^{(0)}$ and the more accurate is the resulting $\hat{\omega}_n$. Quantification of this relationship in the presence of interfering sinusoids and noise is a main objective of this paper.

3 Main Results

We assume that η is a function of n such that $\eta \to 1^-$ as $n \to \infty$. An equivalent assumption is that $\delta := 1 - \eta \to 0^+$. This assumption is necessary in order to achieve the optimal error rate for frequency estimation. Furthermore, for any $k, \ell \in \{1, \cdots, p\}$, let $\omega_{k\ell}^{\pm} := \omega_k \pm \omega_\ell$, $\Delta_{k\ell} := |\omega_{k\ell}^-| = \Delta_{\ell k}$, $\Delta_k := \min\{\Delta_{k\ell} : \ell \neq k\}$, and $\Delta := \min\{\Delta_k : k = 1, \cdots, p\}$. We assume, for analysis purposes, that $\Delta_{k\ell}$, Δ_k, and Δ may depend on n and may tend to zero as $n \to \infty$. This assumption is made in order to model possible frequency clustering (i.e., frequencies that are closely spaced relative to the sample size n). Finally, for technical reasons, we assume that $\{\varepsilon_t\}$ is a martingale difference sequence with respect to some filtration $\{\mathfrak{F}_t\}$ such that $E\{\varepsilon_t^2 | \mathfrak{F}_{t-1}\} = \sigma_\varepsilon^2$ almost surely and $E\{\varepsilon_t^4\} < \infty$ for all t. This assumption is less restrictive than the usual $i.i.d.$ assumption because it can be satisfied as long as the ε_t are independent, but not necessarily identically distributed, with mean zero, variance σ_ε^2, and finite fourth-order moments (in this case $\mathfrak{F}_t$ is the sigma field generated by $\{\varepsilon_\tau, \tau \leq t\}$). Note that a martingale difference sequence is a white (uncorrelated) noise process, due to the fact that $\mathrm{Cov}(\varepsilon_t, \varepsilon_{t-\tau}) = E\{E(\varepsilon_t \varepsilon_{t-\tau} | \mathfrak{F}_{t-1})\} = E\{E(\varepsilon_t | \mathfrak{F}_{t-1}) \varepsilon_{t-\tau}\} = 0$ for all $\tau > 0$. Note also that the complete knowledge about the distribution of $\{\varepsilon_t\}$, which may or may not be Gaussian, is not required in our analysis.

3.1 Asymptotic Properties

This section contains five theorems regarding some asymptotic properties of the CM estimator. The first theorem concerns the existence of the CM estimator $\hat{\alpha}_n$ as a fixed-point of $\rho_n(\alpha)$ and the convergence of the CM iteration (2.5) to $\hat{\alpha}_n$ for a given initial value $\hat{\alpha}_n^{(0)}$.

Theorem 1 *Let $\mathcal{A}_{nk} := \{\alpha : |\alpha - \alpha_k| \le a\delta^{\varepsilon}\} \subset \mathcal{A}$ be a neighborhood of α_k, where $a > 0$ and $\varepsilon \in (1, \frac{3}{2})$ are constants. Assume that as $n \to \infty$, $n\eta^n = \mathcal{O}(1)$, $\delta^{3-2\varepsilon} \log n \to 0$, and $\Delta^{-2}\delta^{2-\varepsilon} = \mathcal{O}(1)$. Then, for sufficiently large n, the mapping $\alpha \mapsto \rho_n(\alpha)$ has almost surely a unique fixed-point $\hat{\alpha}_n$ in $\mathcal{A}_{nk}$ such that $\rho_n(\hat{\alpha}_n) = \hat{\alpha}_n$; and for any $\hat{\alpha}_n^{(0)} \in \mathcal{A}_{nk}$, the probability that the sequence $\{\hat{\alpha}_n^{(m)}\}$ defined by (2.5) converges to $\hat{\alpha}_n$ as $m \to \infty$ is equal to unity.*

Note that $n\eta^n = \mathcal{O}(1)$ implies $\delta n \to \infty$. Therefore, Theorem 1 requires that δ approach zero slower than n^{-1}. On the other hand, it also requires that δ approach zero faster than $(\log n)^{-1/(3-2\varepsilon)}$ so that $\delta^{3-2\varepsilon} \log n \to 0$. Both conditions are satisfied with the choice of $\delta = \mathcal{O}(n^{-\nu})$ for any fixed $\nu \in (0,1)$. For a given δ, Theorem 1 requires that the minimum separation of the frequencies be at least $\mathcal{O}(\delta^{1-\varepsilon/2})$, or $\mathcal{O}(n^{-\nu+\varepsilon\nu/2})$ if $\delta = \mathcal{O}(n^{-\nu})$.

The next theorem shows that the CM estimator is strongly consistent.

Theorem 2 *Assume that the conditions in Theorem 1 are satisfied. Let $\hat{\omega}_n$ be defined by (2.6) where $\hat{\alpha}_n$ is the fixed-point of $\rho_n(\alpha)$ in $\mathcal{A}_{nk}$ obtained from (2.5). Then, for any $d \le \varepsilon$, $\delta^{-d}(\hat{\omega}_n - \omega_k) \to 0$ almost surely as $n \to \infty$. In particular, $\hat{\omega}_n \to \omega_k$ almost surely as $n \to \infty$.*

In practice, the initial values may be provided by another estimation procedure. It is more appropriate in such cases to regard $\hat{\alpha}_n^{(0)}$ as a random variable rather than a constant. For random initial values, Theorems 1 and 2 can be modified as follows.

Theorem 3 *Let the conditions in Theorem 1 be satisfied. For any $\hat{\alpha}_n^{(0)}$, if $P\{\hat{\alpha}_n^{(0)} \in \mathcal{A}_{nk}\} \to 1$ as $n \to \infty$, then, the probability that the sequence $\{\hat{\alpha}_n^{(m)}\}$ converges to $\hat{\alpha}_n$ as $m \to \infty$ approaches unity as $n \to \infty$. Moreover, for any $d \le \varepsilon$, $\delta^{-d}(\hat{\omega}_n - \omega_k) \to 0$ in probability as $n \to \infty$.*

Depending on how quickly δ tends to zero, different rates of weak convergence to normality can be established for the CM estimator. Two useful cases are considered in the following.

Theorem 4 *Assume that the conditions in Theorem 1 are satisfied. If, in addition, $\delta^2 n \to \infty$, $\delta^{5-2r} n = \mathcal{O}(1)$, and $\Delta^{-4}\delta^r \to 0$ for some constant $r \in (0,1]$, then $\delta^{-3/2} n^{1/2} (\hat{\omega}_n - \omega_k - \delta^2 \eta^{-1} b_k^{-1}) \xrightarrow{D} \mathcal{N}(0, \gamma_k^{-2})$ as $n \to \infty$,*

Table 1: Different Scenarios in Theorem 4

Bandwidth	Initial Accuracy	Error of $\hat{\omega}_n$	Δ^{-1}	$C_n(\alpha, \hat{\alpha}_n)$
$\nu = \frac{1}{5}^{+}$ $(r = 0^{+})$	$o_P(n^{-1/5})$ $(\varepsilon = 1^{+})$	$o_P(n^{-2/5})$	$\mathcal{O}(1)$	$o(1)$
$\nu = \frac{1}{3}$ $(r = 1)$	$o_P(n^{-1/3})$ $(\varepsilon = 1^{+})$	$\mathcal{O}_P(n^{-2/3})$	$o(n^{1/12})$	$o(1)$
$\nu \in (\frac{1}{3}, \frac{1}{2})$ $(r = 1)$	$\mathcal{O}_P(n^{-1/2})$ $(\varepsilon = \frac{1}{2\nu})$	$\mathcal{O}_P(n^{-2\nu})$	$o(n^{\nu/4})$	$\mathcal{O}(n^{-1/2+\nu})$

where $\gamma_k := \frac{1}{2}\beta_k^2/\sigma_\varepsilon^2$ *and* $b_k := 2\beta_k^2/\sum_{\ell \neq k}\beta_\ell^2\{\cot(\frac{1}{2}\omega_{k\ell}^{-})) + \cot(\frac{1}{2}\omega_{k\ell}^{+})\}$ *are the signal-to-noise ratio and the signal-to-interference ratio of the kth sinusoid, respectively.*

The requirements in Theorem 4 can be satisfied by $\delta = \mathcal{O}(n^{-\nu})$ for any $\nu \in [\frac{1}{5-2r}, \frac{1}{2})$. According to Theorem 3, the initial precision for the CM iteration to converge can be expressed as $\mathcal{O}_P(n^{-\varepsilon\nu})$. This requirement is satisfied by any estimator whose accuracy is $o_P(n^{-1/(5-2r)})$, which obviously includes all root-n consistent estimators. With such initial values, the iteration (2.5) is guaranteed by Theorem 3 to converge to the desired CM estimator, at least with probability tending to unity. By Theorem 4, the error of $\hat{\omega}_n$ takes the form

$$\hat{\omega}_n - \omega_k = \max\{\mathcal{O}_P(n^{-2\nu}), \mathcal{O}_P(n^{-(1+3\nu)/2})\} = \mathcal{O}_P(n^{-2\nu}),$$

which is always as small as $o_P(n^{-2/5})$ because $\nu \in [\frac{1}{5-2r}, \frac{1}{2}) \subseteq (\frac{1}{5}, \frac{1}{2})$ for any $r \in (0, 1]$. To achieve this error rate, it is required by Theorem 4 that $\Delta^{-1} = o(\delta^{-r/4})$, i.e., the separation of the frequencies be greater than $\mathcal{O}(\delta^{r/4})$. Note that $\Delta^{-4}\delta^{r} \to 0$ for any $r \in (0, 1]$ implies $\Delta^{-2}\delta^{2-\varepsilon} \to 0$. Therefore, the frequency separation condition is stronger in Theorem 4 than in Theorem 1.

Under three different scenerios, Table 1 summarizes the role of bandwidth selection in determining the required initial accuracy, the error rate of the resulting CM estimator, and the required frequency separation. It shows in particular that if a root-n consistent estimator is employed as the initial guess, then the CM iteration is guaranteed to converge and the resulting error rate can be made arbitrarily close to $\mathcal{O}_P(n^{-1})$ by choosing ν near $\frac{1}{2}^{-}$ (the third row in Table 1), provided the frequency separation is greater than $\mathcal{O}(n^{-1/8})$.

Note that due to the interference from other sinusoids, the CM estimator in Theorem 4 is not as precise as it would be in the case of single sinusoid for the same bandwidth [2]. The interference appears in Theorem 4 as the deterministic bias term $\delta^2\eta^{-1}b_k^{-1}$. This term dominates the random error that takes the form $\mathcal{O}(\delta^{3/2}n^{-1/2})$ and thus determines the precision of $\hat{\omega}_n$. Since the bias tends to zero as $n \to \infty$, $\hat{\omega}_n$ remains to be consistent for estimating ω_k, as ensured by Theorem 3. Except the bias, the asymptotic distribution of $\hat{\omega}_n$ in Theorem 4 is the same as in the single-frequency case discussed in [2].

Table 2: Different Scenarios in Theorem 5

Bandwidth	Initial Accuracy	Error of $\hat{\omega}_n$	Δ^{-1}	$C_n(\alpha, \hat{\alpha}_n)$
$\nu = \frac{1}{2}^+$	$o_P(n^{-1/2})$ $(\varepsilon = 1^+)$	$o_P(n^{-1})$	$\mathcal{O}(n^{1/8})$	$o(1)$
$\nu \in (\frac{1}{2}, \frac{2}{3})$	$\mathcal{O}_P(n^{-2/3})$ $(\varepsilon = \frac{2}{3\nu})$	$\mathcal{O}_P(n^{-2\nu})$	$o(n^{\nu/4})$	$\mathcal{O}(n^{-3/2+\nu})$
$\nu = \frac{2}{3}^+$	$o_P(n^{-2/3})$ $(\varepsilon = 1^+)$	$o_P(n^{-4/3})$	$\mathcal{O}(n^{1/6})$	$o(1)$
$\nu \in (\frac{2}{3}, 1)$	$\mathcal{O}_P(n^{-1})$ $(\varepsilon = \frac{1}{\nu})$	$\mathcal{O}_P(n^{-1-\nu/2})$	$o(n^{\nu/4})$	$\mathcal{O}(n^{-1+\nu})$

Theorem 4 requires that δ approach zero at least as fast as $n^{-1/(5-2r)}$ but slower than $n^{-1/2}$. The next theorem concerns two situations in which δ approaches zero faster than $n^{-1/2}$.

Theorem 5 *Let the conditions in Theorem 1 be satisfied.* (a) *If* $\delta^2 n \to 0$ *and* $\Delta^{-4}\delta \to 0$, *then* $\delta^{-1/2} n\,(\hat{\omega}_n - \omega_k - \delta^2 \eta^{-1} b_k^{-1}) \xrightarrow{D} \mathcal{N}(0, \gamma_k^{-1})$ *as* $n \to \infty$. (b) *If* $\delta^{3/2} n \to 0$ *and* $\Delta^{-4}\delta \to 0$, *then* $\delta^{-1/2} n\,(\hat{\omega}_n - \omega_k) \xrightarrow{D} \mathcal{N}(0, \gamma_k^{-1})$ *as* $n \to \infty$.

Again, the asymptotic distribution of $\hat{\omega}_n$ in Theorem 5 is the same as in the single-frequency case discussed in [2], except the interference-induced bias. The conditions in Theorem 5 can be satisfied by $\delta = \mathcal{O}(n^{-\nu})$ for any $\nu \in (\frac{1}{2}, 1)$. By Theorems 1 and 3, the required initial accuracy takes the form $\mathcal{O}(n^{-\varepsilon\nu})$ for almost sure convergence or $\mathcal{O}_P(n^{-\varepsilon\nu})$ for convergence with probability tending to unity. The error of the resulting $\hat{\omega}_n$ can be expressed as $\max\{\mathcal{O}_P(n^{-2\nu}), \mathcal{O}_P(n^{-1-\nu/2})\}$, which implies that

$$\hat{\omega}_n - \omega_k = \begin{cases} \mathcal{O}_P(n^{-2\nu}) & \text{if } \nu \in (\frac{1}{2}, \frac{2}{3}), \\ \mathcal{O}_P(n^{-1-\nu/2}) & \text{if } \nu \in (\frac{2}{3}, 1). \end{cases}$$

Table 2 summarizes several scenarios of bandwidth selection under the conditions in Theorem 5. As can be seen, the error of $\hat{\omega}_n$ is always smaller than $\mathcal{O}_P(n^{-1})$ if $\nu \in (\frac{1}{2}, \frac{2}{3})$ and smaller than $\mathcal{O}_P(n^{-4/3})$ if $\nu \in (\frac{2}{3}, 1)$. Most importantly, by choosing ν near 1^- (the fourth row in Table 2), the error rate can be made arbitrarily close to the optimal value $\mathcal{O}_P(n^{-3/2})$.

3.2 A Three-Step Algorithm

Based on the asymptotic results, we now propose a bandwidth selection rule that capitalizes on the ability of the CM estimator in accommodating poor initial values to produce improved frequency estimates. This leads to a three-step algorithm for achieving the optimal statistical efficiency with a computational complexity comparable to that of FFT.

As shown in Table 2, in order to approach the optimal error rate, the initial guess should be at least as accurate as $\mathcal{O}_P(n^{-1})$. Such an initial guess can be obtained from the CM iteration with any $\nu \in (\frac{1}{2}, \frac{2}{3})$, because Theorem 5 guarantees that the resulting estimator is always more accurate than $\mathcal{O}_P(n^{-1})$. To obtain the latter estimator from the CM iteration, the required initial accuracy is reduced to $o_P(n^{-1/2})$, which, according to Theorem 4, can be satisfied by the CM estimates with any $\nu \in [\frac{1}{3}, \frac{1}{2})$ when initialized by any estimates of precision $o_P(n^{-1/3})$ (see Table 1).

In summary, with three increasing values of ν, namely

$$\nu_1 \in [\tfrac{1}{3}, \tfrac{1}{2}), \quad \nu_2 \in (\tfrac{1}{2}, \tfrac{2}{3}), \quad \nu_3 = 1^-,$$

the CM method is able to improve upon any initial estimates of precision $o_P(n^{-1/3})$ and converge to a final estimate whose accuracy is arbitrarily close to the optimal rate $\mathcal{O}_P(n^{-3/2})$. Note that if a root-n consistent estimator is employed as the first initial guess, then it suffices to take $\nu_1 \in (\frac{1}{3}, \frac{1}{2})$. Note also that the required frequency separation depends on the accuracy of the first initial guess: if that accuracy is merely $o_P(n^{-1/3})$, then the separation should be greater than $\mathcal{O}(n^{-1/12})$; if a root-n consistent estimator is employed as the first initial guess, then it suffices that the separation be greater than $\mathcal{O}(n^{-1/8})$.

The computational complexity of this three-step algorithm is comparable to that of n-point FFT, both taking the form $\mathcal{O}(n \log n)$. To prove this assertion, consider the expressions of the contraction coefficient $C_n(\alpha, \hat{\alpha}_n)$ given in Tables 1 and 2. According to (4.13) and (4.14),

$$\hat{\alpha}_n^{(m+1)} - \hat{\alpha}_n = C_n^{(m)} \, (\hat{\alpha}_n^{(m)} - \hat{\alpha}_n),$$

where $C_n^{(m)} := C_n(\hat{\alpha}_n^{(m)}, \hat{\alpha}_n)$. In Step 3 of the algorithm, we have $C_n^{(m)} = o(1)$ because $\nu_3 = 1^-$ (Row 4 in Table 2). Therefore, the number of iterations required to achieve the desired accuracy $\mathcal{O}_P(n^{-3/2})$ from an initial value of accuracy $\mathcal{O}_P(n^{-1})$ can be expressed as $\mathcal{O}(\log n)$. Similarly, the number of iterations required in Step 2 takes the form $\mathcal{O}(1)$ because $C_n^{(m)} = \mathcal{O}(n^{-3/2+\nu_2})$ (Row 2 in Table 2). In Step 1, if the initial accuracy is $o_P(n^{-1/3})$, then the required number of iterations is $\mathcal{O}(\log n)$ (Row 2 in Table 1); if the initial accuracy is $\mathcal{O}_P(n^{-1/2})$, then that number is reduced to $\mathcal{O}(1)$ (Row 3 in Table 1). Therefore, the total number of iterations required to achieve the optimal error rate from an initial accuracy $o_P(n^{-1/3})$ or $\mathcal{O}_P(n^{-1/2})$ can be expressed as $\mathcal{O}(\log n)$. The overall complexity takes the form $\mathcal{O}(n \log n)$ because the complexity of updating the estimate in each iteration is $\mathcal{O}(n)$.

A simulation example is shown in Figure 1 to demonstrate the algorithm. The time series in this example contains three equal-amplitude sinusoids of frequencies $\omega_1 = 2\pi \times 10.5/n$, $\omega_2 = 2\pi \times 11.5/n$, and $\omega_3 = 2\pi \times 20.5/n$, where $n = 64$. The noise is a zero-mean white Gaussian process, with the sample variance

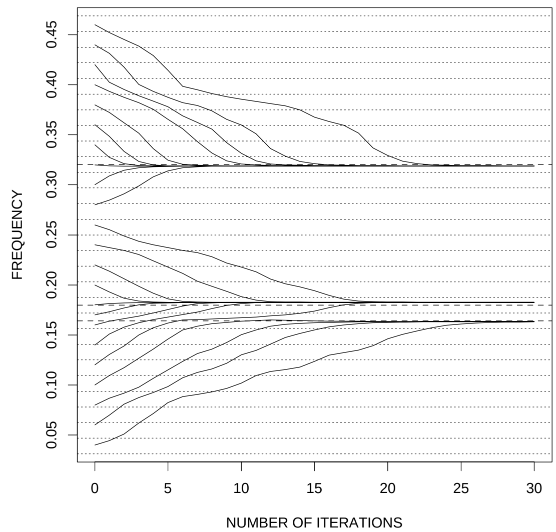

Figure 1: Normalized CM frequency estimates $\hat{\omega}_n^{(m)}/(2\pi)$ versus the number of iterations m for different initial values. Dashed lines represent the true frequencies and dotted lines represent the Fourier frequencies j/n $(j = 1, 2, \cdots)$. The sample size is $n = 64$, and the signal-to-noise ratio is -6 dB for each sinusoid. The bandwidth parameters are $\eta_1 = 0.96$ for $1 \leq m \leq 6$, $\eta_2 = 0.98$ for $7 \leq m \leq 12$, and $\eta_3 = 0.99$ for $m \geq 13$.

standardized so that the signal-to-noise ratio of each sinusoid is equal to -6 dB. Figure 1 shows the trajectory of the normalized frequency estimates $\hat{\omega}_n^{(m)}/(2\pi)$, as functions of m, obtained with different initial values, where $\hat{\omega}_n^{(m)} := \arccos(\hat{\alpha}_n^{(m)})$. The CM iteration begins with $\eta_1 = 0.96$; after 6 iterations, the bandwidth parameter is increased to $\eta_2 = 0.98$, and after 6 additional iterations, it is increased to $\eta_3 = 0.99$. For each initial value, the iteration converges to one of the frequencies with the final (highest) accuracy determined by the last (smallest) bandwidth.

As suggested in [29], the convergence of the CM iteration can be accelerated by replacing $\rho_n(\alpha)$ in (2.5) with the modified mapping $\tilde{\rho}_n(\alpha) := \rho_n(\alpha)\,\mu + \alpha\,(1-\mu)$, where $\mu \neq 1$ is a constant. Note that $\tilde{\rho}_n(\alpha)$ has the same fixed-points as $\rho_n(\alpha)$. Furthermore, since the contraction coefficient of $\tilde{\rho}_n(\alpha)$ is $\tilde{C}_n(\alpha, \hat{\alpha}_n) := 1 - \mu\,\{1 - C_n(\alpha, \hat{\alpha}_n)\}$, the CM iteration with the modified mapping is guaranteed to converge (under the conditions in Theorem 1) if μ satisfies $0 < \mu < 2/\{1 - C_n(\alpha, \hat{\alpha}_n)\}$. The choice of $\mu = 2$ is valid in particular when $C_n(\alpha, \hat{\alpha}_n) > 0$ (as is the case in Figure 1). For $\mu = 2$, $|\tilde{C}_n(\alpha, \hat{\alpha}_n)| < C_n(\alpha, \hat{\alpha}_n)$ if $\frac{1}{3} < C_n(\alpha, \hat{\alpha}_n) < 1$. This means that accelerated convergence can be achieved with $\tilde{\rho}_n(\alpha)$ when the convergence with $\rho_n(\alpha)$ is slow (e.g., $C_n(\alpha, \hat{\alpha}_n) \approx 1$). Figure 2 shows the trajectory of the CM estimates obtained with the modified mapping ($\mu = 2$) for the same data used in Figure 1. Accelerated convergence is evident.

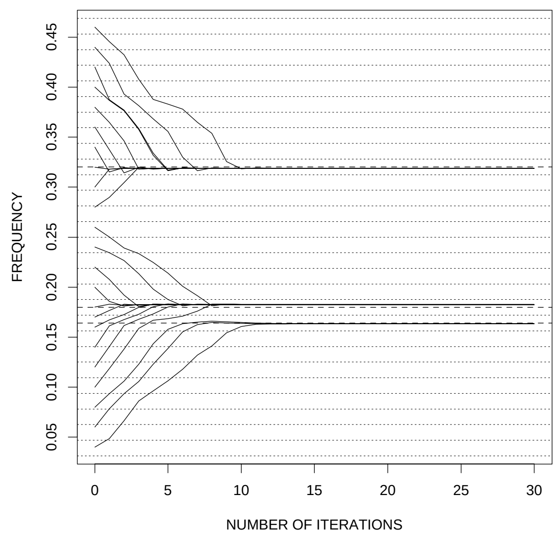

Figure 2: Same as in Figure 1 except that the modified mapping $\tilde{\rho}_n(\alpha)$ with $\mu = 2$ is employed. The bandwidth parameters are $\eta_1 = 0.96$ for $1 \le m \le 4$, $\eta_2 = 0.98$ for $5 \le m \le 8$, and $\eta_3 = 0.99$ for $m \ge 9$.

3.3 Remarks

So far, the ϕ_k in (1.1) are assumed to be constants. Alternatively, the ϕ_k can be 'randomized' by assuming that they are $i.i.d.$ random variables with uniform distribution in $(-\pi, \pi]$ and are independent of $\{\varepsilon_t\}$. This leads to a stochastic signal model in which the sinusoids become (second-order) stationary random processes. The randomization does not alter our results presented in the previous sections because these results do not dependent on the values of ϕ_k. This can be easily justified, as demonstrated in [26], by first conditioning on the ϕ_k to obtain a probabilistic statement (e.g., an estimator exists almost surely or with probability tending to unity, or an estimator converges in distribution to a normal random variable whose mean and variance do not depend on the ϕ_k) and then taking the expected value of the conditional probabilities with respect to the ϕ_k. The same remark applies to the asymptotic error rate of MP and NLS frequency estimators.

Even though our discussion is focused on real-valued sinusoids, similar results can be obtained under the complex-valued sinusoid-plus-noise model $y_t = \sum_{k=1}^{p} \beta_k \exp\{i(\omega_k t + \phi_k)\} + \varepsilon_t$. In this case, it suffices to consider a complex AR(1) filter $(1 + \alpha\mathfrak{B})^{-1}$, where $\alpha := \eta \exp(i\omega)$ and $\eta \in (0,1)$. For this filter, the (ensemble) lag-one autocorrelation coefficient of the filtered (white) noise is equal to α. Therefore, the filter satisfies the "fundamental property" required by the CM method [22]. This property implies that when $p = 1$ the lag-one autocorrelation coefficient of the filtered $\{y_t\}$ forms a contractive mapping whose unique fixed-point is equal to $\eta \exp(i\omega_1)$. Therefore, as in the real-valued case, frequency estimators can be obtained from the sample lag-one autocorrelation of the filtered observations. Note that the parameter η controls the bandwidth of the AR(1) filter in the same way as it does the AR(2) filter.

4 Proof of the Theorems

The theorems are proved in this section on the basis of some preliminary propositions that are proved (if necessary) later in Sec. 5.

First, we introduce some useful notation. Let Λ_{nk} be the set of $\lambda \in (0,\pi)$ determined by (2.3) with $\alpha \in \mathcal{A}_{nk} \subset \mathcal{A}$. Since $\eta \to 1$ and hence $\mathcal{A} \to (-1,1)$ as $n \to \infty$, it follows that α_k becomes an interior point of $\mathcal{A}$ for large n. Furthermore, since the length of $\mathcal{A}_{nk}$ decreases with the increase of n, the interval $\mathcal{A}_{nk}$, there exists a closed subinterval $\mathcal{A}_k$, which is independent of n, such that $\mathcal{A}_{nk} \subset \mathcal{A}_k$ for large n. As a result, there is a closed subinterval $\Lambda_k \subset (0,\pi)$, which is independent of n, such that $\Lambda_{nk} \subset \Lambda_k$ for large n. Therefore, any $\lambda \in \Lambda_{nk}$ can be uniformly bounded away from 0 and π for large n.

Moreover, according to (2.2) and (2.3), we can write

$$y_t(\alpha) + \eta^2 y_{t-2}(\alpha) = y_t + (1+\eta^2)\,\alpha\, y_{t-1}(\alpha).$$

Therefore, with $\lambda \in (0,\pi)$ defined by (2.3), the mapping $\rho_n(\alpha)$ in (2.4) can be expressed as

$$\rho_n(\alpha) = \alpha + (1+\eta^2)^{-1} \sin\lambda\, \frac{\Phi_n(\lambda)}{\Psi_n(\lambda)}, \tag{4.1}$$

where

$$\Psi_n(\lambda) := \sin^2\lambda \sum_{t=1}^{n} y_{t-1}^2(\alpha), \quad \Phi_n(\lambda) := \sin\lambda \sum_{t=1}^{n} y_t\, y_{t-1}(\alpha). \tag{4.2}$$

Equation (4.1) shows that the behavior of $\rho_n(\alpha)$ in a neighborhood of α_k is determined by the behavior of $\Psi_n(\lambda)$ and $\Phi_n(\lambda)$ in a neighborhood of $\lambda_k \in (0,\pi)$, where λ_k is defined by

$$\cos\lambda_k = \frac{1+\eta^2}{2\eta}\,\alpha_k. \tag{4.3}$$

The propositions in the following describe the behavior of $\Psi_n(\lambda)$ and $\Phi_n(\lambda)$ and are prerequisite to the proof of the theorems.

4.1 Preliminary Propositions

The first two propositions describe some asymptotic characteristics of $\Psi_n(\lambda)$ and $\Phi_n(\lambda)$.

Proposition 1 *Let $\Psi_n(\lambda)$ be defined by (4.2) with $\lambda \in \Lambda_{nk}$, $\alpha \in \mathcal{A}_{nk}$, and $\varepsilon \in (1, \frac{3}{2})$. As $n \to \infty$, assume that $\delta \to 0$, $n(1-\delta)^n = \mathcal{O}(1)$, and $\Delta^{-1}\delta \to 0$. Then,*

$$
\begin{aligned}
\Psi_n(\lambda) \;=\; & \tfrac{1}{8}\beta_k^2\eta^{-2}\delta^{-2}n + \mathcal{O}(\Delta_k^{-4}n) + \mathcal{O}(\Delta_k^{-2}\delta^{-1}n) + \mathcal{O}(\Delta_k^{-2}\delta^{-1/2}n\sqrt{\log n}) \\
& + \mathcal{O}(\delta^{-3}) + \mathcal{O}(\delta^{-3/2}n\sqrt{\log n}) + \mathcal{O}(\delta^{-1}n\log n) \\
& + (\lambda-\lambda_k)\{\mathcal{O}(\Delta_k^{-6}\delta^{-1}) + \mathcal{O}(\Delta_k^{-4}\delta^{-2}) + \mathcal{O}(\Delta_k^{-2}\delta^{-2}n) + \mathcal{O}(\delta^{\varepsilon-4}n) \\
& + \mathcal{O}(\delta^{-5/2}n\sqrt{\log n}) + \mathcal{O}(\delta^{-2}n\log n)\}
\end{aligned}
\tag{4.4}
$$

almost surely and uniformly in $\lambda \in \Lambda_{nk}$ for sufficiently large n. Under the same assumptions,

$$
\begin{aligned}
\Psi_n(\lambda) - \Psi_n(\lambda') \;=\; & (\lambda-\lambda')\{\mathcal{O}(\Delta_k^{-6}\delta^{-1}) + \mathcal{O}(\Delta_k^{-4}\delta^{-2}) + \mathcal{O}(\Delta_k^{-2}\delta^{-2}n) + \mathcal{O}(\delta^{\varepsilon-4}n) \\
& + \mathcal{O}(\delta^{-5/2}n\sqrt{\log n}) + \mathcal{O}(\delta^{-2}n\log n)\}
\end{aligned}
\tag{4.5}
$$

almost surely and uniformly in $\lambda, \lambda' \in \Lambda_{nk}$.

Proposition 2 *Let $\Phi_n(\lambda)$ be defined by (4.2). If the conditions in Proposition 1 are satisfied, then*

$$
\begin{aligned}
\Phi_n(\lambda) \;=\; & n\eta^{-1}\xi_k + \mathcal{O}(\Delta_k^{-4}) + \mathcal{O}(\Delta_k^{-2}\delta^{-1}) + \mathcal{O}(\Delta_k^{-4}\delta^2 n) \\
& + \mathcal{O}(\delta^{-1/2}n\sqrt{\log n}) + \mathcal{O}(\delta^{-1}\sqrt{n\log n}) + \mathcal{O}(\delta^{-3/2}\sqrt{\log n}) \\
& + (\lambda-\lambda_k)\{\tfrac{1}{4}\beta_k^2\delta^{-2}n + \mathcal{O}(\Delta_k^{-4}n) + \mathcal{O}(\Delta_k^{-2}\delta^{-2}) + \mathcal{O}(\delta^{-3}) + \mathcal{O}(\delta^{\varepsilon-3}n) \\
& + \mathcal{O}(\delta^{-3/2}n\sqrt{\log n}) + \mathcal{O}(\delta^{-2}\sqrt{n\log n}) + \mathcal{O}(\delta^{-5/2}\sqrt{\log n})\}
\end{aligned}
\tag{4.6}
$$

almost surely and uniformly in $\lambda \in \Lambda_{nk}$ for sufficiently large n, where $\xi_k := \frac{1}{8}\sum_{\ell\neq k}\beta_\ell^2\{\cot(\frac{1}{2}\omega_{k\ell}^-) + \cot(\frac{1}{2}\omega_{k\ell}^+)\}$. Under the same assumptions,

$$
\begin{aligned}
\Phi_n(\lambda) - \Phi_n(\lambda') \;=\; & (\lambda-\lambda')\{\tfrac{1}{4}\beta_k^2\delta^{-2}n + \mathcal{O}(\Delta_k^{-4}n) + \mathcal{O}(\Delta_k^{-2}\delta^{-2}) + \mathcal{O}(\delta^{-3}) + \mathcal{O}(\delta^{\varepsilon-3}n) \\
& + \mathcal{O}(\delta^{-3/2}n\sqrt{\log n}) + \mathcal{O}(\delta^{-2}\sqrt{n\log n}) + \mathcal{O}(\delta^{-5/2}\sqrt{\log n})\}
\end{aligned}
\tag{4.7}
$$

almost surely and uniformly in $\lambda, \lambda' \in \Lambda_{nk}$.

The next result is presented without proof because it can be easily obtained from Propositions 1 and 2 together with the fact that $\lambda - \lambda_k = \mathcal{O}(\delta^{\varepsilon})$ for any $\lambda \in \Lambda_{nk}$.

Corollary 1 *Let the conditions in Proposition 1 be satisfied. If, in addition, $\delta^{3-2\varepsilon}\log n \to 0$ and $\Delta_k^{-2}\delta \to 0$*

as $n \to \infty$, then

$$\begin{aligned}
\Psi_n(\lambda) &= \tfrac{1}{8}\beta_k^2\eta^{-2}\delta^{-2}n\{1+\mathcal{O}(\Delta_k^{-2}\delta)+\mathcal{O}(\delta^{-1}n^{-1})+\mathcal{O}(\delta^{1/2}\sqrt{\log n})+\mathcal{O}(\delta^{2\varepsilon-2})\},\\
\Psi_n(\lambda_k) &= \tfrac{1}{8}\beta_k^2\eta^{-2}\delta^{-2}n\{1+\mathcal{O}(\Delta_k^{-2}\delta)+\mathcal{O}(\delta^{-1}n^{-1})+\mathcal{O}(\delta^{1/2}\sqrt{\log n})\},\\
\Psi_n(\lambda)\Psi_n(\lambda') &= \tfrac{1}{64}\beta_k^4\eta^{-4}\delta^{-4}n^2\{1+\mathcal{O}(\Delta_k^{-2}\delta)+\mathcal{O}(\delta^{-1}n^{-1})+\mathcal{O}(\delta^{1/2}\sqrt{\log n})+\mathcal{O}(\delta^{2\varepsilon-2})\},\\
\Psi_n(\lambda)-\Psi_n(\lambda') &= (\lambda-\lambda')\{\mathcal{O}(\Delta_k^{-2}\delta^{-2}n)+\mathcal{O}(\delta^{\varepsilon-4}n)\},\\
\Phi_n(\lambda) &= \mathcal{O}(\Delta_k^{-2}\delta^{-1})+\mathcal{O}(\delta^{\varepsilon-2}n),\\
\Phi_n(\lambda_k) &= \mathcal{O}(\Delta_k^{-2}\delta^{-1})+\mathcal{O}(\delta^{-1/2}n\sqrt{\log n}),\\
\Phi_n(\lambda)-\Phi_n(\lambda') &= (\lambda-\lambda')\tfrac{1}{4}\beta_k^2\delta^{-2}n\{1+\mathcal{O}(\Delta_k^{-4}\delta^2)+\mathcal{O}(\delta^{-1}n^{-1})+\mathcal{O}(\delta^{1/2}\sqrt{\log n})+\mathcal{O}(\delta^{\varepsilon-1})\},
\end{aligned}$$

almost surely and uniformly in λ, $\lambda' \in \Lambda_{nk}$ for sufficiently large n.

The next two propositions play an important role in establishing the asymptotic normality of the CM frequency estimator. One of them is cited from the literature without proof.

Proposition 3 *Under the conditions in Proposition 1,*

$$\begin{aligned}
\Phi_n(\lambda_k) &= W_{n1}+W_{n2}+n\eta^{-1}\xi_k+\mathcal{O}_P(\Delta_k^{-4})+\mathcal{O}_P(\Delta_k^{-2}\delta^{-1})\\
&\quad +\mathcal{O}_P(\Delta_k^{-4}\delta^2 n)+\mathcal{O}_P(\Delta_k^{-2}n^{1/2}),
\end{aligned}$$

where

$$W_{n1} := \tfrac{1}{2}\beta_k\delta\, g(\lambda_k-\omega_k)\sum_{t=1}^{n}\varepsilon_t\,(\eta^{n-t}-\eta^{t-1})\sin(t\omega_k+\phi_k), \tag{4.8}$$

$$W_{n2} := \sum_{t=1}^{n}\sum_{j=0}^{t-1}\eta^{j-1}\sin(j\lambda_k)\,\varepsilon_t\,\varepsilon_{t-j}, \tag{4.9}$$

and $g(\lambda) := (1-2\eta\cos\lambda+\eta^2)^{-1}$.

Proposition 4 [2] *Let W_{n1} and W_{n2} be defined by (4.8) and (4.9), respectively. Then, under the conditions in Proposition 1, $\delta^{3/2}W_{n1} \xrightarrow{D} \mathcal{N}(0, \frac{1}{8}\beta_k^2\sigma_\varepsilon^2)$ and $\delta^{1/2}n^{-1/2}W_{n2} \xrightarrow{D} \mathcal{N}(0, \frac{1}{4}\sigma_\varepsilon^4)$ as $n \to \infty$.*

The final proposition, cited without proof, describes some useful relations between λ and α.

Proposition 5 [27] *Let $\mathcal{A}_{nk}$ be defined in Theorem 1 with $\alpha_k = \cos\omega_k$ and $\varepsilon > 0$. If $\eta \to 1$ ($\delta = 1-\eta \to 0$) as $n \to \infty$, then the following assertions are valid.*

(a) *Let* λ_k *be defined by* (4.3), *then*

$$\lambda_k - \omega_k = -\tfrac{1}{2}\eta^{-1}\delta^2\alpha_k(1-\alpha_k^2)^{-1/2} + \mathcal{O}(\delta^4).$$

Moreover, there exist constants $c_0 > 0$ *and* $n_0 > 0$ *such that* $|\lambda_k - \omega_k| \le c_0\delta^2$ *for* $n > n_0$.

(b) *Let* $\lambda, \lambda' \in \Lambda_{nk}$ *be determined by* $\alpha, \alpha' \in \mathcal{A}_{nk}$ *according to* (2.3), *then*

$$\lambda - \lambda' = -\frac{1}{\sin\lambda'}\,\frac{1+\eta^2}{2\eta}\,(\alpha-\alpha')\left\{1 + \frac{\xi\sin\lambda'}{2(1-\xi^2)^{3/2}}\,\frac{1+\eta^2}{2\eta}\,(\alpha-\alpha')\right\}$$

where $\xi \in (-1,1)$ *depends on* λ *and* λ' *and there exist constants* $0 < c < 1$ *and* $n_0 > 0$ *such that* $\xi^2 \le c$ *for all* $\lambda, \lambda' \in \Lambda_{nk}$ *and for* $n > n_0$.

Equipped with these propositions, let us now prove the theorems.

4.2 Proof of Theorem 1

It suffices to show that $\rho_n(\alpha)$ is a contractive mapping in $\mathcal{A}_{nk}$. This can be done by proving that the following inequalities hold almost surely for sufficiently large n (e.g., [30], p. 251, Theorem 5.2.3):

$$|\rho_n(\alpha) - \rho_n(\alpha')| \le c\,|\alpha - \alpha'| \tag{4.10}$$

for all $\alpha, \alpha' \in \mathcal{A}_{nk}$, where $c \in (0,1)$ is a constant, and

$$|\rho_n(\alpha_k) - \alpha_k| \le (1-c)\,a\delta^{\varepsilon}, \tag{4.11}$$

where c is given in (4.10) and a is given in Theorem 1. Let us now prove these inequalities.

Proof of (4.10). It follows from (4.1) that

$$\rho_n(\alpha) - \rho_n(\alpha') = \alpha - \alpha' + R_n, \tag{4.12}$$

where

$$\begin{aligned}
R_n &:= (J_1 + J_2 + J_3)\{(1+\eta^2)\Psi_n(\lambda)\Psi_n(\lambda')\}^{-1},\\
J_1 &:= \{\Psi_n(\lambda') - \Psi_n(\lambda)\}\Phi_n(\lambda)\sin\lambda,\\
J_2 &:= \{\Phi_n(\lambda) - \Phi_n(\lambda')\}\Psi_n(\lambda)\sin\lambda,\\
J_3 &:= (\sin\lambda - \sin\lambda')\Phi_n(\lambda')\Psi_n(\lambda).
\end{aligned}$$

Note that $\Delta^{-2}\delta^{2-\varepsilon} = \mathcal{O}(1)$ implies $\Delta^{-2}\delta \to 0$. Therefore, by Corollary 1,

$$\begin{aligned} J_1 &= (\lambda-\lambda')\{\mathcal{O}(\Delta_k^{-4}\delta^{-3}n)+\mathcal{O}(\Delta_k^{-2}\delta^{\varepsilon-5}n)+\mathcal{O}(\Delta_k^{-2}\delta^{\varepsilon-4}n^2)+\mathcal{O}(\delta^{2\varepsilon-6}n^2)\}, \\ J_2 &= (\lambda-\lambda')\sin\lambda\,\tfrac{1}{32}\beta_k^4\eta^{-2}\delta^{-4}n^2\{1+\mathcal{O}(\Delta_k^{-2}\delta)+\mathcal{O}(\delta^{-1}n^{-1})+\mathcal{O}(\delta^{1/2}\sqrt{\log n})+\mathcal{O}(\delta^{\varepsilon-1})\}, \\ J_3 &= (\lambda-\lambda')\{\mathcal{O}(\Delta_k^{-2}\delta^{-3}n)+\mathcal{O}(\delta^{\varepsilon-4}n^2)\}\}. \end{aligned}$$

Since $\sin\lambda$ can be bounded away from zero uniformly for all $\lambda \in \Lambda_{nk}$, it follows that

$$\begin{aligned} J_1+J_2+J_3 &= (\lambda-\lambda')\sin\lambda\,\tfrac{1}{32}\beta_k^4\eta^{-2}\delta^{-4}n^2\{1+\mathcal{O}(\Delta_k^{-2}\delta) \\ &\quad +\mathcal{O}(\delta^{-1}n^{-1})+\mathcal{O}(\delta^{1/2}\sqrt{\log n})+\mathcal{O}(\delta^{\varepsilon-1})\}. \end{aligned}$$

This, combined with the expression for $\Psi_n(\lambda)\Psi_n(\lambda')$ in Corollary 1, leads to

$$\begin{aligned} R_n &= (\lambda-\lambda')\sin\lambda\,2\eta^2(1+\eta^2)^{-1}\{1+\mathcal{O}(\Delta_k^{-2}\delta) \\ &\quad +\mathcal{O}(\delta^{-1}n^{-1})+\mathcal{O}(\delta^{1/2}\sqrt{\log n})+\mathcal{O}(\delta^{\varepsilon-1})\}. \end{aligned}$$

Furthermore, since $\alpha-\alpha' = \mathcal{O}(\delta^{\varepsilon})$, it follows from Proposition 5(b) that $\lambda-\lambda' = -(2\eta\sin\lambda)^{-1}(1+\eta^2)(\alpha-\alpha')\{1+\mathcal{O}(\delta^{\varepsilon})\}$. Substituting this expression in the foregoing equation yields

$$R_n = -\eta\,(\alpha-\alpha')\{1+\mathcal{O}(\Delta_k^{-2}\delta)+\mathcal{O}(\delta^{-1}n^{-1})+\mathcal{O}(\delta^{1/2}\sqrt{\log n})+\mathcal{O}(\delta^{\varepsilon-1})\}.$$

Therefore, (4.12) can be rewritten as

$$\rho_n(\alpha)-\rho_n(\alpha') = C_n(\alpha,\alpha')(\alpha-\alpha'), \tag{4.13}$$

where $C_n(\alpha,\alpha') := \{\rho_n(\alpha)-\rho_n(\alpha')\}/(\alpha-\alpha')$ can be expressed as

$$\begin{aligned} C_n(\alpha,\alpha') &= 1-\eta\{1+\mathcal{O}(\Delta_k^{-2}\delta)+\mathcal{O}(\delta^{-1}n^{-1})+\mathcal{O}(\delta^{1/2}\sqrt{\log n})+\mathcal{O}(\delta^{\varepsilon-1})\} \\ &= \delta+\mathcal{O}(\Delta_k^{-2}\delta)+\mathcal{O}(\delta^{-1}n^{-1})+\mathcal{O}(\delta^{1/2}\sqrt{\log n})+\mathcal{O}(\delta^{\varepsilon-1}). \end{aligned} \tag{4.14}$$

The proof is complete upon noting that $C_n(\alpha,\alpha') \overset{a.s.}{\to} 0$ uniformly in α, $\alpha' \in \mathcal{A}_{nk}$.

Proof of (4.11). It follows from (4.1) that $\rho_n(\alpha_k)-\alpha_k = \sin\lambda_k\,\Phi_n(\lambda_k)\{(1+\eta^2)\Psi_n(\lambda_k)\}^{-1}$, where λ_k is defined by (4.3). According to Corollary 1,

$$\frac{\Phi_n(\lambda_k)}{\Psi_n(\lambda_k)} = \mathcal{O}(\Delta_k^{-2}\delta n^{-1})+\mathcal{O}(\delta^{3/2}\sqrt{\log n}).$$

Since $\delta^{3-2\varepsilon}\log n \to 0$, one can write $\mathcal{O}(\delta^{3/2}\sqrt{\log n}) = o(\delta^{\varepsilon})$. Since $\delta n \to \infty$ and $\Delta^{-2}\delta^{2-\varepsilon} = \mathcal{O}(1)$, one can write $\Delta_k^{-2}\delta^{1-\varepsilon}n^{-1} \to 0$ and hence $\Delta_k^{-2}\delta n^{-1} = o(\delta^{\varepsilon})$. Combining these results yields

$$\rho_n(\alpha_k)-\alpha_k = o(\delta^{\varepsilon})$$

almost surely for large n. The proof is thus complete.

4.3 Proof of Theorem 2

Let $\hat{\alpha}_n$ be the fixed-point of $\rho_n(\alpha)$ in $\mathcal{A}_{nk}$. Then, it follows from (4.13) that

$$\hat{\alpha}_n - \alpha_k = C_{nk}\,(\hat{\alpha}_n - \alpha_k) + \rho_n(\alpha_k) - \alpha_k,$$

where $C_{nk} := C_n(\hat{\alpha}_n, \alpha_k)$. This equation can be rewritten as

$$\hat{\alpha}_n - \alpha_k = (1 - C_{nk})^{-1}\{\rho_n(\alpha_k) - \alpha_k\},$$

which, combined with (4.1), leads to

$$\hat{\alpha}_n - \alpha_k = \{(1+\eta^2)\,(1 - C_{nk})\}^{-1} \sin\lambda_k \frac{\Phi_n(\lambda_k)}{\Psi_n(\lambda_k)}. \tag{4.15}$$

Moreover, by Corollary 1, $\delta^2 n^{-1}\Psi_n(\lambda_k) \to \frac{1}{8}\beta_k^2$ and $\Phi_n(\lambda_k) = \mathcal{O}(\Delta_k^{-2}\delta^{-1}) + \mathcal{O}(\delta^{-1/2} n\sqrt{\log n})$ almost surely as $n \to \infty$. Combining these results with the fact that $C_{nk} \to 0$ gives rise to $\hat{\alpha}_n - \alpha_k = \mathcal{O}(\delta^2 n^{-1}\Phi_n(\lambda_k))$ $= \mathcal{O}(\Delta_k^{-2}\delta n^{-1}) + \mathcal{O}(\delta^{3/2}\sqrt{\log n})$. Finally. since $\delta^{3-2\varepsilon}\log n \to 0$, $\delta n \to \infty$, and $\Delta^{-2}\delta^{2-\varepsilon} = \mathcal{O}(1)$, it follows that

$$\delta^{-d}(\hat{\alpha}_n - \alpha_k) = \mathcal{O}(\Delta_k^{-2}\delta^{1-d} n^{-1}) + \mathcal{O}(\delta^{3/2-d}\sqrt{\log n}) \stackrel{a.s.}{\to} 0$$

for any $d \le \varepsilon$. The proof is complete upon noting (2.6) and the fact that $\omega_k = \arccos(\alpha_k)$.

4.4 Proof of Theorem 3

If $\rho_n(\alpha)$ is contractive (in $\mathcal{A}_{nk}$), i.e., if it satisfies (4.10) and (4.11), then $\hat{\alpha}_n^{(m)} \to \hat{\alpha}_n$ as $m \to \infty$ for any $\hat{\alpha}_n^{(0)} \in \mathcal{A}_{nk}$. This implies that

$$\begin{aligned}
&P\Big\{\lim_{m\to\infty} \hat{\alpha}_n^{(m)} = \hat{\alpha}_n\Big\} \\
&\quad\ge\quad P\{\rho_n(\alpha) \text{ is contractive and } \hat{\alpha}_n^{(0)} \in \mathcal{A}_{nk}\} \\
&\quad=\quad P\{\rho_n(\alpha) \text{ is contractive}\} \\
&\qquad\quad - P\{\rho_n(\alpha) \text{ is contractive and } \hat{\alpha}_n^{(0)} \notin \mathcal{A}_{nk}\} \\
&\quad\ge\quad P\{\rho_n(\alpha) \text{ is contractive}\} - P\{\hat{\alpha}_n^{(0)} \notin \mathcal{A}_{nk}\}.
\end{aligned}$$

By Theorem 1, $P\{\rho_n(\alpha) \text{ is contractive}\} = 1$ for large n. By assumption, $P\{\hat{\alpha}_n^{(0)} \in \mathcal{A}_{nk}\} \to 1$ as $n \to \infty$. Combining these results leads to $P\{\lim_{m\to\infty} \hat{\alpha}_n^{(m)} = \hat{\alpha}_n\} \to 1$ as $n \to \infty$. The second part of the assertion follows from a similar argument coupled with Theorem 2.

4.5 Proof of Theorem 4

Consider (4.15), and observe that $\sin\lambda_k \stackrel{a.s.}{\to} \sin\omega_k$ by Proposition 5(a), $\delta^2 n^{-1}\Psi_n(\lambda_k) \stackrel{a.s.}{\to} \frac{1}{8}\beta_k^2$ by Corollary 1, and $C_{nk} \stackrel{a.s.}{\to} 0$ by (4.14). Therefore, according to Slutsky's theorem (e.g., [31], p. 337, Theorem 1.4), $\delta^{-3/2}n^{1/2}(\hat{\alpha}_n - \alpha_k)$ has the same asymptotic distribution as

$$Z_{n1} := 4\beta_k^{-2}\sin\omega_k\,\delta^{1/2}n^{-1/2}\Phi_n(\lambda_k). \tag{4.16}$$

Note that $\Phi_n(\lambda_k)$ has the expression in Proposition 3 where $W_{n1} = \mathcal{O}_P(\delta^{-3/2})$ by Proposition 4. Therefore, under the assumption that $\delta^2 n \to \infty$, $\delta^{5-2r}n = \mathcal{O}(1)$, and $\Delta^{-4}\delta^r \to 0$, it follows from Proposition 3 that $\delta^{1/2}n^{-1/2}\{\Phi_n(\lambda_k) - n\eta^{-1}\xi_k\}$ has the same asymptotic distribution as $\delta^{1/2}n^{-1/2}W_{n2}$, which, by Proposition 4, is $\mathcal{N}(0, \frac{1}{4}\sigma_\varepsilon^4)$. Therefore,

$$Z_{n1} - \delta^{1/2}n^{1/2}\eta^{-1}\xi_k' \stackrel{D}{\to} \mathcal{N}(0, \gamma_k^{-2}\sin^2\omega_k),$$

where $\xi_k' := 4\beta_k^{-2}\xi_k\sin\omega_k$. Combining these results yields

$$\delta^{-3/2}n^{1/2}(\hat{\alpha}_n - \alpha_k - \delta^2\eta^{-1}\xi_k') \stackrel{D}{\to} \mathcal{N}(0, \gamma_k^{-2}\sin^2\omega_k).$$

Since $\xi_k' = b_k^{-1}\sin\omega_k$, the proof is complete upon noting that $\hat{\omega}_n - \omega_k$ has the same asymptotic distribution as $(\hat{\alpha}_n - \alpha_k)/\sin\omega_k$ by the delta method (e.g., [31], p. 337, Theorem 1.5).

4.6 Proof of Theorem 5

By using an argument similar to the proof of Theorem 4, one can show from (4.15) that $\delta^{-1/2}n(\hat{\alpha}_n - \alpha_k)$ has the same asymptotic distribution as

$$Z_{n2} := 4\beta_k^{-2}\sin\omega_k\,\delta^{3/2}\Phi_n(\lambda_k),$$

where $\Phi_n(\lambda_k)$ has the expression in Proposition 3 and $W_{n2} = \mathcal{O}_P(\delta^{-1/2}n^{1/2})$ by Proposition 4. Therefore, under the assumption that $\delta^2 n \to 0$ and $\Delta^{-4}\delta \to 0$, it follows from Proposition 3 that $\delta^{3/2}\{\Phi_n(\lambda_k) - n\eta^{-1}\xi_k\}$ has the same asymptotic distribution as $\delta^{3/2}W_{n1}$, namely $\mathcal{N}(0, \frac{1}{8}\beta_k^2\sigma_\varepsilon^2)$ by Proposition 4. This implies that

$$Z_{n2} - \delta^{3/2}n\eta^{-1}\xi_k' \stackrel{D}{\to} \mathcal{N}(0, \gamma_k^{-1}\sin^2\omega_k),$$

and hence

$$\delta^{-1/2}n(\hat{\alpha}_n - \alpha_k - \delta^2\eta^{-1}\xi_k') \stackrel{D}{\to} \mathcal{N}(0, \gamma_k^{-1}\sin^2\omega_k).$$

An application of the delta method leads to $\delta^{-1/2}n(\hat{\omega}_n - \omega_k - \delta^2\eta^{-1}b_k^{-1}) \stackrel{D}{\to} \mathcal{N}(0, \gamma_k^{-1})$. Furthermore, if, in addition, $\delta^{3/2}n \to 0$, then $\delta^{-1/2}n(\hat{\omega}_n - \omega_k)$ has the same asymptotic distribution $\mathcal{N}(0, \gamma_k^{-1})$ because $\delta^{-1/2}n \times \delta^2\eta^{-1}b_k^{-1} \to 0$. The proof is complete.

5 Proof of the Propositions

This section is devoted to the proof of Propositions 1–3. For convenience, the following short-hand notation will be employed throughout the section:

$$c_k(t) := \beta_k \cos(t\omega_k + \phi_k), \quad s_k(t) := \beta_k \sin(t\omega_k + \phi_k), \tag{5.1}$$

$$p_t(\lambda) := \sum_{j=0}^{t-1} \eta^{j-1} \sin(j\lambda), \quad q_t(\lambda) := \sum_{j=0}^{t-1} \eta^{j-1} \cos(j\lambda), \tag{5.2}$$

$$f_{t\ell}(\lambda) := \sum_{j=0}^{t-1} \eta^{j-1} \sin(j\lambda)\cos(j\omega_\ell), \quad g_{t\ell}(\lambda) := \sum_{j=0}^{t-1} \eta^{j-1} \sin(j\lambda)\sin(j\omega_\ell). \tag{5.3}$$

Lemmas 4–8 in Sec. 6 contain the essential technical results regarding these and some related quantities. It is also important to note that $y_t(\alpha)$, which is defined by (2.2) and (2.3), can be written explicitly as

$$y_t(\alpha) = \frac{1}{\sin\lambda} \sum_{j=0}^{t} \eta^{j-1} \sin(j\lambda) y_{t-j+1}. \tag{5.4}$$

This expression can be verified simply by substituting (5.4) into the left-hand side of (2.2) and confirming that the substitution results in y_t, which is the right-hand side of (2.2). Finally, it is alway assumed in the sequel that $\lambda \in \Lambda_{nk}$ and $\varepsilon \in (1, \frac{3}{2})$.

5.1 Proof of Proposition 1

Let $z_k(t) := \sum_{\ell \neq k} c_\ell(t)$. In estimating the kth frequency, $z_k(t)$ can be regarded as the interference from the other sinusoids. By replacing y_t in (5.4) with its definition given by (1.1), we can write

$$y_{t-1}(\alpha) = \frac{1}{\sin\lambda}\{u_t(\lambda) + v_t(\lambda) + w_t(\lambda)\}, \tag{5.5}$$

where

$$u_t(\lambda) := \sum_{j=0}^{t-1} \eta^{j-1} \sin(j\lambda) c_k(t-j), \tag{5.6}$$

$$v_t(\lambda) := \sum_{j=0}^{t-1} \eta^{j-1} \sin(j\lambda) \varepsilon_{t-j}, \tag{5.7}$$

$$w_t(\lambda) := \sum_{j=0}^{t-1} \eta^{j-1} \sin(j\lambda) z_k(t-j). \tag{5.8}$$

Note that $u_t(\lambda)$ and $v_t(\lambda)$ represent the contributions of the kth sinusoid and the noise, respectively; these terms remain the same as in the case of single sinusoid. The third term $w_t(\lambda)$ is the contribution of the other sinusoids; it is the extra term in the case of multiple sinusoids.

To prove (4.4) and (4.5) with $\Psi_n(\lambda)$ defined by (4.2), we first note that

$$\begin{aligned} y_{t-1}^2(\alpha) &= \frac{1}{\sin^2\lambda}\{u_t^2(\lambda)+v_t^2(\lambda)+2\,u_t(\lambda)\,v_t(\lambda) \\ &\quad + w_t^2(\lambda)+2\,u_t(\lambda)\,w_t(\lambda)+2\,v_t(\lambda)\,w_t(\lambda)\}. \end{aligned} \tag{5.9}$$

As can be seen, the first three terms in (5.9) are the same as in the case of single sinusoid. The remaining terms involve the contribution of the other sinusoids. It suffices to show that these terms are asymptotically negligible. The main effort in the following proof is to evaluate the contribution, $w_t(\lambda)$, of the interfering sinusoids. With this in mind, we now prove (4.4) and (4.5).

5.1.1 Proof of (4.4)

By substituting (5.9) in the expression of $\Psi_n(\lambda)$ in (4.2), we can write

$$\begin{aligned} \Psi_n(\lambda) &= V_n(\lambda)+\sum_{t=1}^{n}\{w_t^2(\lambda)+2\,u_t(\lambda)\,w_t(\lambda)+2\,v_t(\lambda)\,w_t(\lambda)\} \\ &:= V_n(\lambda)+T_1(\lambda)+T_2(\lambda)+T_3(\lambda), \end{aligned} \tag{5.10}$$

where $V_n(\lambda) := \sum\{u_t^2(\lambda)+2\,u_t(\lambda)\,v_t(\lambda)+v_t^2(\lambda)\}$. Since $V_n(\lambda)$ is the same as in the case of single sinusoid, by Proposition 1 in [2], $V_n(\lambda)$ has the asymptotic expression in the right-hand side of (4.4) with $\Delta_k := 1$. Therefore, it suffices to show that the following expressions hold almost surely and uniformly in $\lambda \in \Lambda_{nk}$ for large n:

$$T_1(\lambda) = \mathcal{O}(\Delta_k^{-4}n)+(\lambda-\lambda_k)\{\mathcal{O}(\Delta_k^{-2}\delta^{-2}n)+\mathcal{O}(\delta^{\varepsilon-4}n)\}, \tag{5.11}$$

$$\begin{aligned} T_2(\lambda) &= \mathcal{O}(\Delta_k^{-2}\delta^{-1}n)+(\lambda-\lambda_k)\{\mathcal{O}(\Delta_k^{-6}\delta^{-1})+\mathcal{O}(\Delta_k^{-4}\delta^{-2}) \\ &\quad + \mathcal{O}(\Delta_k^{-2}\delta^{-2}n)+\mathcal{O}(\delta^{\varepsilon-4}n)\}, \end{aligned} \tag{5.12}$$

$$T_3(\lambda) = \mathcal{O}(\Delta_k^{-2}\delta^{-1/2}n\sqrt{\log n})+(\lambda-\lambda_k)\,\mathcal{O}(\delta^{-5/2}n\sqrt{\log n}). \tag{5.13}$$

These expressions are derived in the following by an intensive use of Taylor series expansion (TSE).

A. Proof of (5.11)

The TSE of $T_1(\lambda) := \sum w_t^2(\lambda)$ at λ_k takes the form

$$T_1(\lambda) = T_1(\lambda_k)+(\lambda-\lambda_k)\,\dot{T}_1(\lambda^*), \tag{5.14}$$

where λ^* lies between λ and λ_k, and $\dot{T}_1(\lambda)$ is the derivative of $T_1(\lambda)$.

First, consider $T_1(\lambda_k)$ in (5.14). It is easy to show from (5.8) that

$$w_t(\lambda) = \sum_{\ell \neq k} \{c_\ell(t)\, f_{t\ell}(\lambda) + s_\ell(t)\, g_{t\ell}(\lambda)\} := w_{t1}(\lambda) + w_{t2}(\lambda). \tag{5.15}$$

Therefore, one can write

$$T_1(\lambda_k) = \sum_{t=1}^{n} \{w_{t1}^2(\lambda_k) + 2\, w_{t1}(\lambda_k)\, w_{t2}(\lambda_k) + w_{t2}^2(\lambda_k)\}. \tag{5.16}$$

Since $\lambda_k - \omega_k = \mathcal{O}(\delta^2)$ by Proposition 5(a), it follows from Lemma 6(a) and Lemma 7(b) that $f_{t\ell}(\lambda_k) = \mathcal{O}(\Delta_{\ell k}^{-2}) + \mathcal{O}(1) = \mathcal{O}(\Delta_{\ell k}^{-2})$. Similarly, $g_{t\ell}(\lambda_k) = \mathcal{O}(\Delta_{\ell k}^{-2})$. Substituting these results in (5.15) yields

$$w_{t1}(\lambda_k) = \mathcal{O}(\Delta_k^{-2}), \quad w_{t2}(\lambda_k) = \mathcal{O}(\Delta_k^{-2}), \quad w_t(\lambda_k) = \mathcal{O}(\Delta_k^{-2}). \tag{5.17}$$

This, combined with (5.16), implies that

$$T_1(\lambda_k) = \mathcal{O}(\Delta_k^{-4} n). \tag{5.18}$$

Furthermore, it is easy to see from (5.16) that

$$\begin{aligned} \dot{T}_1(\lambda^*) &= 2\sum_{t=1}^{n} (\dot{w}_{t1}^*\, w_{t1}^* + \dot{w}_{t1}^*\, w_{t2}^* + \dot{w}_{t2}^*\, w_{t1}^* + \dot{w}_{t2}^*\, w_{t2}^*) \\ &:= T_{11} + T_{12} + T_{13} + T_{14}, \end{aligned}$$

where $w_{tj}^* := w_{tj}(\lambda^*)$ and $\dot{w}_{tj}^* := \dot{w}_{tj}(\lambda^*)$ $(j = 1, 2)$. The first term $T_{11} := 2\sum \dot{w}_{t1}^*\, w_{t1}^*$ can be rewritten as

$$T_{11} = 2 \sum_{\ell, \ell' \neq k} \sum_{t=1}^{n} c_\ell(t)\, c_{\ell'}(t)\, f_{t\ell}(\lambda^*)\, \dot{f}_{t\ell'}(\lambda^*).$$

Since $\lambda^* - \omega_k = \mathcal{O}(\delta^\varepsilon)$, it follows from Lemma 6(a) and Lemma 7(b) that $f_{t\ell}(\lambda^*) = \mathcal{O}(\Delta_{\ell k}^{-2}) + \mathcal{O}(\delta^{\varepsilon-2})$. By Lemma 7(a), $\dot{f}_{t\ell'}(\lambda^*) = \mathcal{O}(\delta^{-2})$. Thus, $f_{t\ell}(\lambda^*)\, \dot{f}_{t\ell'}(\lambda^*) = \mathcal{O}(\Delta_{\ell k}^{-2}\delta^{-2}) + \mathcal{O}(\delta^{\varepsilon-4})$ and $T_{11} = \mathcal{O}(\Delta_k^{-2}\delta^{-2} n) + \mathcal{O}(\delta^{\varepsilon-4} n)$. Similarly, $T_{12} := 2\sum \dot{w}_{t1}^*\, w_{t2}^*$ can be rewritten as

$$T_{12} = 2 \sum_{\ell, \ell' \neq k} \sum_{t=1}^{n} s_\ell(t)\, c_{\ell'}(t)\, g_{t\ell}(\lambda^*)\, \dot{f}_{t\ell'}(\lambda^*).$$

Since $g_{t\ell}(\lambda^*) = \mathcal{O}(\Delta_{\ell k}^{-2}) + \mathcal{O}(\delta^{\varepsilon-2})$, it follows that $g_{t\ell}(\lambda^*)\, \dot{f}_{t\ell'}(\lambda^*) = \mathcal{O}(\Delta_{\ell k}^{-2}\delta^{-2}) + \mathcal{O}(\delta^{\varepsilon-4})$ and $T_{12} = \mathcal{O}(\Delta_k^{-2}\delta^{-2} n) + \mathcal{O}(\delta^{\varepsilon-4} n)$. A similar argument can be employed to show that T_{13} and T_{14} also take the form $\mathcal{O}(\Delta_k^{-2}\delta^{-2} n) + \mathcal{O}(\delta^{\varepsilon-4} n)$. Combining these results yields

$$\dot{T}_1(\lambda^*) = \mathcal{O}(\Delta_k^{-2}\delta^{-2} n) + \mathcal{O}(\delta^{\varepsilon-4} n). \tag{5.19}$$

The proof is complete after substituting (5.19) and (5.18) into (5.14).

B. Proof of (5.12)

By definition, $T_2(\lambda) = 2\sum u_t(\lambda)\, w_t(\lambda)$. Consider the TSE

$$T_2(\lambda) = T_2(\lambda_k) + \dot{T}_2(\lambda^*)\,(\lambda - \lambda_k), \tag{5.20}$$

where λ^* is between λ and λ_k. It is easy to show from (5.6) that

$$u_t(\lambda) = c_k(t)\, f_{tk}(\lambda) + s_k(t)\, g_{tk}(\lambda). \tag{5.21}$$

Thus, by Lemma 7(a), $u_t(\lambda_k) = \mathcal{O}(\delta^{-1})$. This, combined with (5.17), leads to $u_t(\lambda_k)\, w_t(\lambda_k) = \mathcal{O}(\Delta_k^{-2}\delta^{-1})$, which, in turn, implies that

$$T_2(\lambda_k) = \mathcal{O}(\Delta_k^{-2}\delta^{-1} n). \tag{5.22}$$

Moreover, it is easy to see that

$$\dot{T}_2(\lambda^*) = 2\sum_{t=1}^{n}\{\dot{u}_t(\lambda^*)\, w_t(\lambda^*) + u_t(\lambda^*)\, \dot{w}_t(\lambda^*)\} := \dot{T}_{21} + \dot{T}_{22}.$$

Therefore, if one can show that

$$\dot{T}_2(\lambda^*) = \mathcal{O}(\Delta_k^{-6}\delta^{-1}) + \mathcal{O}(\Delta_k^{-4}\delta^{-2}) + \mathcal{O}(\Delta_k^{-2}\delta^{-2} n) + \mathcal{O}(\delta^{\varepsilon-4} n), \tag{5.23}$$

then (5.12) would be obtained by substituting (5.23) and (5.22) into (5.20).

To prove (5.23), we note that $\dot{u}_t(\lambda^*) = \mathcal{O}(\delta^{-2})$ by Lemma 7(a) and $w_t(\lambda^*) = \mathcal{O}(\Delta_k^{-2}) + \mathcal{O}(\delta^{\varepsilon-2})$ by Lemma 6(a) and Lemma 7(c). Therefore, $\dot{T}_{21} := 2\sum \dot{u}_t(\lambda^*)\, w_t(\lambda^*) = \mathcal{O}(\Delta_k^{-2}\delta^{-2} n) + \mathcal{O}(\delta^{\varepsilon-4} n)$. To evaluate $\dot{T}_{22} := 2\sum u_t(\lambda^*)\, \dot{w}_t(\lambda^*)$, we note that by Lemma 7(b), $u_t(\lambda^*) = u_t(\lambda_k) + \mathcal{O}(\delta^{\varepsilon-2})$ and $\dot{w}_t(\lambda^*) = \dot{w}_t(\lambda_k) + \mathcal{O}(\delta^{\varepsilon-3})$, and by Lemma 7(a), $u_t(\lambda_k) = \mathcal{O}(\delta^{-1})$ and $\dot{w}_t(\lambda_k) = \mathcal{O}(\delta^{-2})$. Therefore,

$$\begin{aligned} u_t(\lambda^*)\, \dot{w}_t(\lambda^*) &= u_t(\lambda_k)\, \dot{w}_t(\lambda_k) + \mathcal{O}(\delta^{\varepsilon-4}) + \mathcal{O}(\delta^{2\varepsilon-5}) \\ &= u_t(\lambda_k)\, \dot{w}_t(\lambda_k) + \mathcal{O}(\delta^{\varepsilon-4}). \end{aligned}$$

It can be shown, by Lemma 8(c), that $\sum u_t(\lambda_k)\, \dot{w}_t(\lambda_k) = \mathcal{O}(\Delta_k^{-6}\delta^{-1}) + \mathcal{O}(\Delta_k^{-4}\delta^{-2}) + \mathcal{O}(\Delta_k^{-2}\delta^{-3}) + \mathcal{O}(\Delta_k^{-4} n)$ $+\ \mathcal{O}(\delta^{-2} n)$. Therefore, $\dot{T}_{22} = \mathcal{O}(\Delta_k^{-6}\delta^{-1}) + \mathcal{O}(\Delta_k^{-4}\delta^{-2}) + \mathcal{O}(\Delta_k^{-2}\delta^{-3}) + \mathcal{O}(\Delta_k^{-4} n) + \mathcal{O}(\delta^{\varepsilon-4} n)$. This, combined with the fact that $\delta n \to \infty$, proves (5.23).

C. Proof of (5.13)

Consider $T_3(\lambda) := 2\sum v_t(\lambda)\, w_t(\lambda) = T_3(\lambda_k) + \dot{T}_3(\lambda^*)\,(\lambda^* - \lambda_k)$. It follows from Lemma 1(a) that $v_t(\lambda) = \mathcal{O}(\delta^{-1/2}\sqrt{\log t})$ almost surely and uniformly (in λ and η) for large t. This, combined with (5.17), leads to $T_3(\lambda_k) = \mathcal{O}(\Delta_k^{-2}\delta^{-1/2} n\sqrt{\log n})$. Moreover, by Lemma 1(a), $\dot{v}_t(\lambda) = \mathcal{O}(\delta^{-3/2}\sqrt{\log t})$ almost surely and uniformly for large t. Combining these results with $w_t(\lambda^*) = \mathcal{O}(\delta^{-1})$ and $\dot{w}_t(\lambda^*) = \mathcal{O}(\delta^{-2})$, as guaranteed by Lemma 7(a), yields

$$\begin{aligned}
\dot{T}_3(\lambda^*) &= 2\sum_{t=1}^{n}\{v_t(\lambda^*)\,\dot{w}_t(\lambda^*) + \dot{v}_t(\lambda^*)\,w_t(\lambda^*)\} \\
&= \mathcal{O}(\delta^{-2}n)\times\mathcal{O}(\delta^{-1/2}\sqrt{\log n}) + \mathcal{O}(\delta^{-1}n)\times\mathcal{O}(\delta^{-3/2}\sqrt{\log n}) \\
&= \mathcal{O}(\delta^{-5/2}n\sqrt{\log n}). \qquad (5.24)
\end{aligned}$$

Equation (5.13) follows immediately.

5.1.2 Proof of (4.5)

Let $\Psi_n'(\lambda) := T_1(\lambda) + T_2(\lambda) + T_3(\lambda)$. Then, $\Psi_n(\lambda) = V_n(\lambda) + \Psi_n'(\lambda)$. According to Proposition 1 in [2], $V_n(\lambda) - V_n(\lambda')$ has the same asymptotic expression as in the right-hand side of (4.5) with $\Delta_k := 1$. Furthermore, the TSE of $\Psi_n'(\lambda)$ at λ' can be written as

$$\Psi_n'(\lambda) - \Psi_n'(\lambda') = (\lambda - \lambda')\,\{\dot{T}_1(\lambda^*) + \dot{T}_2(\lambda^*) + \dot{T}_3(\lambda^*)\},$$

where λ^* lies between λ and λ'. This, combined with (5.19), (5.23), and (5.24), leads to $\Psi_n'(\lambda) - \Psi_n'(\lambda') = (\lambda - \lambda')\,\{\mathcal{O}(\Delta_k^{-6}\delta^{-1}) + \mathcal{O}(\Delta_k^{-4}\delta^{-2}) + \mathcal{O}(\Delta_k^{-2}\delta^{-2}n) + \mathcal{O}(\delta^{\varepsilon-4}n) + \mathcal{O}(\delta^{-5/2}n\sqrt{\log n})\}$. The proof is complete.

5.2 Proof of Proposition 2

It is easy to show from (1.1), (4.2), and (5.4) that

$$\Phi_n(\lambda) = \sum_{t=1}^{n}\sum_{j=0}^{t-1}\eta^{j-1}\sin(j\lambda)\,y_t\,y_{t-j} = U_n(\lambda) + \sum_{i=1}^{5} S_i(\lambda), \qquad (5.25)$$

where

$$U_n(\lambda) := \sum_{t=1}^{n}\sum_{j=0}^{t-1}\eta^{j-1}\sin(j\lambda)\,\{c_k(t) + \varepsilon_t\}\,\{c_k(t-j) + \varepsilon_{t-j}\}, \qquad (5.26)$$

$$S_1(\lambda) := \sum_{t=1}^{n}\sum_{j=0}^{t-1}\eta^{j-1}\sin(j\lambda)\,z_k(t)\,c_k(t-j) \qquad (5.27)$$

$$S_2(\lambda) := \sum_{t=1}^{n}\sum_{j=0}^{t-1}\eta^{j-1}\sin(j\lambda)\,z_k(t-j)\,\varepsilon_t, \qquad (5.28)$$

$$
\begin{aligned}
S_3(\lambda) &:= \sum_{t=1}^{n}\sum_{j=0}^{t-1} \eta^{j-1}\sin(j\lambda)\, z_k(t-j)\, c_k(t), \\
S_4(\lambda) &:= \sum_{t=1}^{n}\sum_{j=0}^{t-1} \eta^{j-1}\sin(j\lambda)\, z_k(t)\, \varepsilon_{t-j}, \\
S_5(\lambda) &:= \sum_{t=1}^{n}\sum_{j=0}^{t-1} \eta^{j-1}\sin(j\lambda)\, z_k(t)\, z_k(t-j). \qquad (5.29)
\end{aligned}
$$

In these expressions, $n^{-1}S_1(\lambda)$ is the sample covariance between the kth filtered sinusoid and the interference; $n^{-1}S_2(\lambda)$ is the sample covariance between the noise and the filtered interference; $n^{-1}S_3(\lambda)$ is the sample covariance between the kth unfiltered sinusoid and the filtered interference; $n^{-1}S_4(\lambda)$ is the sample covariance between the filtered noise and the unfiltered interference; and $n^{-1}S_5(\lambda)$ is the covariance between the unfiltered and the filtered interferences. With this in mind, let us prove (4.6) and (4.7).

5.2.1 Proof of (4.6)

By Proposition 2 in [2], $U_n(\lambda)$ has the same asymptotic expression as in the right-hand side of (4.6) with $\Delta_k := 1$. Therefore, (4.6) is a direct consequence of the following:

$$
\begin{aligned}
S_1(\lambda) &= \mathcal{O}(\Delta_k^{-4}) + \mathcal{O}(\Delta_k^{-2}\delta^{-1}) \\
&\quad + (\lambda-\lambda_k)\{\mathcal{O}(\Delta_k^{-4}\delta^{-1}) + \mathcal{O}(\Delta_k^{-2}\delta^{-2}) + \mathcal{O}(\delta^{\varepsilon-3}n)\}, \qquad (5.30) \\
S_2(\lambda) &= \mathcal{O}(\delta^{-1}\sqrt{n\log n}) + \mathcal{O}(\delta^{-3/2}\sqrt{\log n}) \\
&\quad + (\lambda-\lambda_k)\{\mathcal{O}(\delta^{-2}\sqrt{n\log n}) + \mathcal{O}(\delta^{-5/2}\sqrt{\log n})\}, \qquad (5.31) \\
S_3(\lambda) &= \text{same as the right-hand side of (5.30)}, \qquad (5.32) \\
S_4(\lambda) &= \text{same as the right-hand side of (5.31)}, \qquad (5.33) \\
S_5(\lambda) &= n\eta^{-1}\xi_k + \mathcal{O}(\Delta_k^{-4}\delta^2 n) + \mathcal{O}(\Delta_k^{-2}\delta^{-1}) \\
&\quad + (\lambda-\lambda_k)\{\mathcal{O}(\Delta_k^{-2}\delta^{-2}) + \mathcal{O}(\Delta_k^{-4}n) + \mathcal{O}(\delta^{\varepsilon-3}n)\}. \qquad (5.34)
\end{aligned}
$$

where ξ_k is defined in Proposition 2. Note that we only need to prove (5.30), (5.31), and (5.34) because (5.32) and (5.33) can be easily derived from these results by observing the symmetry in their definitions.

A. Proof of (5.30)

It is easy to see from (5.27) that

$$
S_1(\lambda) = \sum_{\ell\neq k}\sum_{t=1}^{n} c_\ell(t)\,\{c_k(t)\, f_{tk}(\lambda) + s_k(t)\, g_{tk}(\lambda)\}.
$$

Consider the second-order TSE

$$S_1(\lambda) = S_1(\lambda_k) + (\lambda - \lambda_k)\{\dot{S}_1(\lambda_k) + \tfrac{1}{2}\ddot{S}_1(\lambda^*)(\lambda - \lambda_k)\}, \tag{5.35}$$

where λ^* is between λ and λ_k. By Lemma 8(a) and Lemma 8(b), $S_1(\lambda_k) = \mathcal{O}(\Delta_k^{-4}) + \mathcal{O}(\Delta_k^{-2}\delta^{-1})$ and $\dot{S}_1(\lambda_k) = \mathcal{O}(\Delta_k^{-4}\delta^{-1}) + \mathcal{O}(\Delta_k^{-2}\delta^{-2}) + \mathcal{O}(\delta^{-1}n)$. By Lemma 7(a), $\ddot{S}_1(\lambda^*) = \mathcal{O}(\delta^{-3}n)$. Equation (5.30) follows from these results and the fact that $\lambda - \lambda_k = \mathcal{O}(\delta^{\varepsilon})$.

B. Proof of (5.31)

It follows from (5.28) that

$$S_2(\lambda) = \sum_{\ell \neq k}\sum_{t=1}^{n} \varepsilon_t \{c_\ell(t)\, f_{t\ell}(\lambda) + s_\ell(t)\, g_{t\ell}(\lambda)\}. \tag{5.36}$$

Expanding $S_2(\lambda)$ at λ_k gives rise to

$$S_2(\lambda) = S_2(\lambda_k) + (\lambda - \lambda_k)\{\dot{S}_2(\lambda_k) + \tfrac{1}{2}\ddot{S}_2(\lambda^*)(\lambda - \lambda_k)\}, \tag{5.37}$$

where λ^* is between λ and λ_k. It is easy to show, by using Lemma 1(b), that

$$\begin{aligned}
S_2(\lambda_k) &= \mathcal{O}(\delta^{-1}\sqrt{n\log n}) + \mathcal{O}(\delta^{-3/2}\sqrt{\log n}),\\
\dot{S}_2(\lambda_k) &= \mathcal{O}(\delta^{-2}\sqrt{n\log n}) + \mathcal{O}(\delta^{-5/2}\sqrt{\log n}),\\
\ddot{S}_2(\lambda^*) &= \mathcal{O}(\delta^{-3}\sqrt{n\log n}) + \mathcal{O}(\delta^{-7/2}\sqrt{\log n}).
\end{aligned}$$

Equation (5.31) can be proved by substituting these expressions into (5.37) together with the fact that $\lambda - \lambda_k = \mathcal{O}(\delta^{\varepsilon})$, $\mathcal{O}(\delta^{-2}\sqrt{n\log n}) + \mathcal{O}(\delta^{\varepsilon-3}\sqrt{n\log n}) = \mathcal{O}(\delta^{-2}\sqrt{n\log n})$, $\mathcal{O}(\delta^{-5/2}\sqrt{\log n}) + \mathcal{O}(\delta^{\varepsilon-7/2}\sqrt{\log n}) = \mathcal{O}(\delta^{-5/2}\sqrt{\log n})$.

C. Proof of (5.34)

It is easy to see from (5.29) that

$$\begin{aligned}
S_5(\lambda) &= \sum_{\ell,\ell' \neq k}\sum_{t=1}^{n} c_{\ell'}(t)\{c_\ell(t)\, f_{t\ell}(\lambda) + s_\ell(t)\, g_{t\ell}(\lambda)\}\\
&= S_5(\lambda_k) + (\lambda - \lambda_k)\{\dot{S}_5(\lambda_k) + \tfrac{1}{2}\ddot{S}_5(\lambda^*)(\lambda - \lambda_k)\}.
\end{aligned}$$

By Lemma 8(a)-(b), $S_5(\lambda_k) = n\eta^{-1}\xi_k + \mathcal{O}(\Delta_k^{-4}\delta^2 n) + \mathcal{O}(\Delta_k^{-2}\delta^{-1})$ and $\dot{S}_5(\lambda_k) = \mathcal{O}(\Delta_k^{-2}\delta^{-2}) + \mathcal{O}(\Delta_k^{-4}n) + \mathcal{O}(\delta^{-1}n)$. By Lemma 7(a), $\ddot{S}_5(\lambda^*) = \mathcal{O}(\delta^{-3}n)$. The proof is complete upon noting that $\lambda - \lambda_k = \mathcal{O}(\delta^{\varepsilon})$.

5.2.2 Proof of (4.7)

Let $\Phi_n'(\lambda) := \sum_{i=1}^{5} S_i(\lambda)$, so that $\Phi_n(\lambda) = U_n(\lambda) + \Phi_n'(\lambda)$. The TSE of $\Phi_n'(\lambda)$ at λ' can be expressed as $\Phi_n'(\lambda) - \Phi_n'(\lambda') = (\lambda - \lambda')\sum_{i=1}^{5} \dot{S}_i(\lambda^*)$, where λ^* is between λ and λ'. It follows from the proof of (4.6) that $\dot{S}_1(\lambda^*) = \dot{S}_1(\lambda_k) + (\lambda^* - \lambda_k)\ddot{S}_1(\lambda^{**}) = \mathcal{O}(\Delta_k^{-4}\delta^{-1}) + \mathcal{O}(\Delta_k^{-2}\delta^{-2}) + \mathcal{O}(\delta^{\varepsilon-3}n)$, $\dot{S}_2(\lambda^*) = \mathcal{O}(\delta^{-2}\sqrt{n\log n}) + \mathcal{O}(\delta^{-5/2}\sqrt{\log n})$, and $\dot{S}_5(\lambda^*) = \mathcal{O}(\Delta_k^{-2}\delta^{-2}) + \mathcal{O}(\Delta_k^{-4}n) + \mathcal{O}(\delta^{\varepsilon-3}n)$. Moreover, $\dot{S}_3(\lambda^*)$ has the same expression as $\dot{S}_1(\lambda^*)$ and $\dot{S}_4(\lambda^*)$ has the same expression as $\dot{S}_2(\lambda^*)$. The proof is complete upon noting that $U_n(\lambda) - U_n(\lambda')$, by Proposition 2 in [2], has the same asymptotic expression as in the right-hand side of (4.7) with $\Delta_k := 1$.

5.3 Proof of Proposition 3

Consider (5.25). According to Proposition 3 in [2],

$$U_n(\lambda_k) = W_{n1} + W_{n2} + \mathcal{O}_P(\delta^2 n) + \mathcal{O}_P(n^{1/2}) + \mathcal{O}_P(\delta^{-1}).$$

Therefore, it suffices to show that

$$S_i(\lambda_k) = \begin{cases} \mathcal{O}_P(\Delta_k^{-4}) + \mathcal{O}_P(\Delta_k^{-2}\delta^{-1}) & \text{if } i = 1,3, \\ \mathcal{O}_P(\Delta_k^{-2}n^{1/2}) + \mathcal{O}_P(\Delta_k^{-2}\delta^{-1/2}) & \text{if } i = 2,4, \\ n\eta^{-1}\xi_k + \mathcal{O}_P(\Delta_k^{-4}\delta^2 n) + \mathcal{O}_P(\Delta_k^{-2}\delta^{-1}) & \text{if } i = 5. \end{cases}$$

To this end, let $\lambda_{k\ell}^{\pm} := \lambda_k \pm \omega_\ell$. Then, by Lemma 4(d), together with (6.1) and (6.5), both $f_{t\ell}(\lambda_k)$ and $g_{t\ell}(\lambda_k)$ can be expressed as $g(\lambda_{k\ell}^{\pm})$ multiplied by a linear combination of 1 and $\eta^t \exp(\pm it\lambda_{k\ell}^{\pm})$. By Lemma 4(a)-(b), $g(\lambda_{k\ell}^{-}) = \mathcal{O}(\Delta_{\ell k}^{-2})$ and $g(\lambda_{k\ell}^{+}) = \mathcal{O}(1)$. Combining these results with (5.36) and Lemma 1(c) yields the expression for $S_2(\lambda_k)$. Because of the symmetry, $S_4(\lambda_k)$ has the same expression. Furthermore, the expressions for $S_1(\lambda_k)$, $S_3(\lambda_k)$, and $S_5(\lambda_k)$ can be obtained from (5.30), (5.32), and (5.34), respectively. The proof is complete.

6 Technical Lemmas

This section contains some technical results needed in the proof of the propositions. The first three lemmas are cited without proof.

Lemma 1 [27] *As $n \to \infty$, the following expressions are true.*

(a) $\max_\omega |\sum_{t=1}^{n} t^r \eta^t \varepsilon_t \exp(it\omega)| = \mathcal{O}(\delta^{-r-1/2}\sqrt{\log n})$ *almost surely for any integer $r \geq 0$.*

(b) $|\sum_{t=1}^{n}\varepsilon_t \exp(it\omega)\sum_{j=1}^{t} j^r \eta^j \exp(ij\omega')| = \mathcal{O}(\delta^{-r-1}\sqrt{n\log n}) + \mathcal{O}(\delta^{-r-3/2}\sqrt{\log n})$ *almost surely and uniformly in* ω *and* ω' *for* $r = 0,1,2$.

(c) $\sum_{t=1}^{n}\varepsilon_t \exp(it\omega) = \mathcal{O}_P(n^{1/2})$ *and* $\max_{1\le t\le n}|\sum_{j=1}^{t} j^r \eta^j \varepsilon_j \sin(j\omega)| = \mathcal{O}_P(\delta^{-r-1/2})$ *for any given* ω *and any integer* $r \ge 0$.

Lemma 2 [27] *Let* $\mathcal{A}_{nk}$ *be defined in Theorem 1 with* $\alpha_k = \cos\omega_k$ *and* $\varepsilon > 0$. *If* $\eta \to 1$ *as* $n \to \infty$, *then there exist constants* $c_2 > c_1 > 0$ *and* $n_0 > 0$ *such that* $c_1|\alpha - \alpha_k| \le |\lambda - \lambda_k| \le c_2|\alpha - \alpha_k|$ *for all* $\lambda \in \Lambda_{nk}$ *and* $n > n_0$; *in particular, these inequalities imply that* $\lambda - \lambda_k = \mathcal{O}(\delta^{\varepsilon})$ *uniformly for* $\lambda \in \Lambda_{nk}$ *and for sufficiently large n.*

Lemma 3 [27] *Let* $r = 0,1,2$.

(a) $\sum_{t=1}^{n} t^r \eta^t \exp(it\omega) = \mathcal{O}(\delta^{-r-1})$ *uniformly in* $\omega \in (-\infty,\infty)$, $\eta \in (0,1)$, *and* $n > 0$.

(b) *If* ω *is uniformly bounded away from* $2\pi k$ *for any integer k, then* $\sum_{t=1}^{n}\exp(it\omega) = \mathcal{O}(1)$. *If, in addition,* $n\eta^n = \mathcal{O}(1)$, *then* $\sum_{t=1}^{n} t^r \eta^t \exp(it\omega) = \mathcal{O}(1)$.

It is easy to show from (5.2) that

$$p_t(\lambda) = g(\lambda)\, p_{0t}(\lambda), \quad q_t(\lambda) = g(\lambda)\, q_{0t}(\lambda), \tag{6.1}$$

where

$$g(\lambda) := (1 - 2\eta\cos\lambda + \eta^2)^{-1}, \tag{6.2}$$

$$p_{0t}(\lambda) := \sin\lambda - \eta^{t-1}\sin(t\lambda) + \eta^t \sin((t-1)\lambda), \tag{6.3}$$

$$q_{0t}(\lambda) := \eta^{-1}\{1 - \eta\cos\lambda - \eta^t\cos(t\lambda) + \eta^{t+1}\cos((t-1)\lambda)\}. \tag{6.4}$$

These quantities are evaluated in the next lemma.

Lemma 4 *Let* $g(\lambda)$, $p_{0t}(\lambda)$, *and* $q_{0t}(\lambda)$ *be defined by* (6.2)–(6.4). *Then, the following results are true for* $\lambda \in (-2\pi, 2\pi)$ *and* $r = 0,1,2$.

(a) *If* λ *is bounded away from zero and* $\pm 2\pi$, *then* $g(\lambda) = \frac{1}{4}\eta^{-1}(\sin\frac{1}{2}\lambda)^{-2} + \mathcal{O}(\delta^2)$ *and* $g^{(r)}(\lambda) = \mathcal{O}(1)$ *uniformly in* η *and* λ.

(b) *If* $\lambda \to 0$ *and* $\lambda\delta^{-1} \to \infty$ *as* $n \to \infty$, *then* $g(\lambda) = \frac{1}{4}\eta^{-1}(\sin\frac{1}{2}\lambda)^{-2} + \mathcal{O}(\lambda^{-4}\delta^2)$ *and* $g^{(r)}(\lambda) = \mathcal{O}(\lambda^{-2(r+1)})$ *uniformly in* η *and* λ.

(c) *If λ satisfies the condition in* (a) *or* (b), *and if $n\eta^n = \mathcal{O}(1)$, then $\sum_{t=1}^{n} t^r \eta^t \exp(it\lambda) = \mathcal{O}(g^{r+1}(\lambda)) = \mathcal{O}(\lambda^{-2(r+1)})$.*

(d) *As functions of t, $p_{0t}^{(r)}(\lambda)$ and $q_{0t}^{(r)}(\lambda)$ can be regarded as linear combinations of 1 and $t^j \eta^t \exp(\pm it\lambda)$ $(0 \leq j \leq r)$ in which the coefficients are uniformly bounded functions of η and λ.*

Proof. Let $a := 4\eta(\sin\frac{1}{2}\lambda)^2$ and $x := \delta^2$. Then, the TSE of $g(\lambda) = (a+x)^{-1}$ at $x = 0$ can be expressed as $g(\lambda) = a^{-1} - (a+x^*)^{-2}x$, where x^* is between zero and x. If λ is bounded away from zero and $\pm 2\pi$, then $a + x \geq a + x^* \geq c > 0$ for some constant c. This implies that $(a+x^*)^{-2} = \mathcal{O}(1)$ and $g(\lambda) = \mathcal{O}(1)$. The assertion follows immediately upon noting that $\dot{g}(\lambda) = -2\eta g^2(\lambda)\sin\lambda$ and $\ddot{g}(\lambda) = -2\eta g^2(\lambda)\cos\lambda + 8\eta^2 g^3(\lambda)\sin\lambda\cos\lambda$.

To prove part (b), consider the TSE $\cos\lambda = 1 - \frac{1}{2}\lambda^2\cos\lambda^*$, where λ^* is between zero and λ. Since $\lambda \to 0$, there exists a constant $c > 0$ such that $\cos\lambda^* \geq c$ for large n. This implies that $a = 2\eta(1-\cos\lambda) \geq c\eta\lambda^2$. As a result, $a + x^* \geq \lambda^2(x^*/\lambda^2 + c\eta)$. This, combined with the assumption that $x/\lambda^2 \to 0$, leads to $(a+x^*)^{-2} = \mathcal{O}(\lambda^{-4})$, and thus the expression for $g(\lambda)$. Furthermore, since $a + x \geq \lambda^2(x/\lambda^2 + c\eta)$, we obtain $g(\lambda) = \mathcal{O}(\lambda^{-2})$. Part (b) follows immediately. Part (c) follows from Lemma 4 in [27], and part (d) follows from (6.3) and (6.4). Q.E.D.

Equipped with Lemma 4, the following results can be easily obtained.

Lemma 5 *Let $p_t(\lambda)$ and $q_t(\lambda)$ be defined by* (5.2) *or* (6.1).

(a) *If $\lambda \in (-2\pi, 2\pi)$ is bounded away from zero and $\pm 2\pi$, then $p_t^{(r)}(\lambda)$ and $q_t^{(r)}(\lambda)$ $(r = 0,1,2)$ can be regarded as linear combinations of 1 and $t^j\eta^t \exp(\pm it\lambda)$ $(0 \leq j \leq r)$ in which the coefficients are uniformly bounded functions of η and λ.*

(b) *If $\lambda \to 0$ and $\lambda\delta^{-1} \to \infty$ as $n \to \infty$, then $p_t^{(r)}(\lambda)$ and $q_t^{(r)}(\lambda)$ $(r = 0,1,2)$ can be regarded as linear combinations of $\lambda^{-2(r-j+1)}$ and $\lambda^{-2(r-j+1)}t^j\eta^t \exp(\pm it\lambda)$ $(0 \leq j \leq r)$ in which the coefficients are uniformly bounded functions of η and λ.*

(c) *For $r = 0,1$, $p_t^{(r)}(0) = r\delta^{-2}\{1 - \delta t\eta^{t-1} - \eta^t\}$ and $q_t^{(r)}(0) = (1-r)\delta^{-1}(1-\eta^t)\eta^{-1}$.*

Proof. Part (a) follows from (6.1) and Lemma 4(a) and (d). Part (b) results from (6.1) and Lemma 4(b) and (d). Part (c) can be obtained from (5.2) by direct evaluation. Q.E.D.

It is easy to show from (5.3) that

$$f_{t\ell}(\lambda) = \tfrac{1}{2}\{p_t(\lambda - \omega_\ell) + p_t(\lambda + \omega_\ell)\}, \quad g_{t\ell}(\lambda) = \tfrac{1}{2}\{q_t(\lambda - \omega_\ell) - q_t(\lambda + \omega_\ell)\}. \tag{6.5}$$

Therefore, the next lemma, concerning $f_{t\ell}(\lambda)$ and $g_{t\ell}(\lambda)$, is a direct consequence of Lemma 5.

Lemma 6 *Let $f_{t\ell}(\lambda)$ and $g_{t\ell}(\lambda)$ be defined by* (5.3)*. Then, the following statements are true.*

(a) *For $r = 0,1,2$ and $\ell \neq k$, if $\Delta_{\ell k}^{-1}\delta \to 0$ as $n \to \infty$, then $f_{t\ell}^{(r)}(\omega_k)$ and $g_{t\ell}^{(r)}(\omega_k)$ can be considered as linear combinations of 1, $t^j\eta^t\exp(\pm it\omega_{k\ell}^{+})$, $\Delta_{\ell k}^{-2(r-j+1)}$, and $\Delta_{\ell k}^{-2(r-j+1)}t^j\eta^t\exp(\pm it\omega_{k\ell}^{-})$ $(0 \le j \le r)$, in which the coefficients are uniformly bounded functions of η and n. Both $f_{t\ell}^{(r)}(\omega_k)$ and $g_{t\ell}^{(r)}(\omega_k)$ can be expressed as $\mathcal{O}(\Delta_{\ell k}^{-2(r+1)}) + \sum_{j=0}^{r}\mathcal{O}(\Delta_{\ell k}^{-2(r-j+1)}t^j\eta^t)$, which holds uniformly in t and η.*

(b) *For $r = 0,1$, $f_{tk}^{(r)}(\omega_k)$ can be regarded as a linear combination of 1, $r\delta^{-2}$, $r\delta^{-2+j}t^j\eta^t$, and $t^j\eta^t\times$ $\exp(\pm it2\omega_k)$, and $g_{tk}^{(r)}(\omega_k)$ can be regarded as a linear combination of 1, $(1-r)\delta^{-1}$, $(1-r)\delta^{-1}\eta^t$, and $t^j\eta^t\exp(\pm it2\omega_k)$ $(j = 0,r)$, in which the coefficients are uniformly bounded functions of η and n. In general, for $r = 0,1,2$,*

$$\begin{aligned} f_{tk}^{(r)}(\omega_k) &= \tfrac{1}{2}p_t^{(r)}(0) + \mathcal{O}(1) + \sum_{j=0}^{r}\mathcal{O}(t^j\eta^t), \\ g_{tk}^{(r)}(\omega_k) &= \tfrac{1}{2}q_t^{(r)}(0) + \mathcal{O}(1) + \sum_{j=0}^{r}\mathcal{O}(t^j\eta^t), \end{aligned}$$

uniformly in t and η.

Proof. By (6.5), $f_{t\ell}^{(r)}(\omega_k) = \frac{1}{2}\{p_t^{(r)}(\omega_{k\ell}^{-}) + p_t^{(r)}(\omega_{k\ell}^{+})\}$. Lemma 5(a) is applicable to $p_t^{(r)}(\omega_{k\ell}^{+})$ for any ℓ. In the case of $\ell \neq k$, $p_t^{(r)}(\omega_{k\ell}^{-})$ can be evaluated by Lemma 5(a) if $\Delta_{\ell k} = \mathcal{O}(1)$ and by Lemma 5(b) if $\Delta_{\ell k} \to 0$. Lemma 5(c) is applicable in the case of $\ell = k$ and $r = 0,1$. Combining these results leads to the expression for $f_{t\ell}^{(r)}(\omega_k)$. The proof for $g_{t\ell}^{(r)}(\omega_k)$ is similar. Q.E.D.

The next two lemmas are instrumental to the proof of the propositions.

Lemma 7 *Assume that the conditions in Proposition 1 be satisfied. Let $c_k(t)$ and $s_k(t)$ be defined by* (5.1)*, and let $f_{t\ell}(\lambda)$ and $g_{t\ell}(\lambda)$ be defined by* (5.3)*. Then, the following expressions are true.*

(a) *For $r = 0,1,2$ and for any $\ell = 1,\cdots,p$,*

$$f_{t\ell}^{(r)}(\lambda) = \mathcal{O}(\delta^{-r-1}), \quad g_{t\ell}^{(r)}(\lambda) = \mathcal{O}(\delta^{-r-1}),$$

uniformly in t, λ, and η.

(b) *For $r = 0,1,2$, if $\lambda - \omega_k = \mathcal{O}(\delta^q)$ for some $q > 0$, then*

$$f_{t\ell}^{(r)}(\lambda) = f_{t\ell}^{(r)}(\omega_k) + \mathcal{O}(\delta^{q-r-2}), \quad g_{t\ell}^{(r)}(\lambda) = g_{t\ell}^{(r)}(\omega_k) + \mathcal{O}(\delta^{q-r-2}),$$

uniformly in t, η, and λ.

(c) *For $\ell \neq k$ and $\lambda \in \Lambda_{nk}$, if $\Delta_{\ell k}^{-1}\delta \to 0$ as $n \to \infty$, then $f_{t\ell}(\lambda)$ and $g_{t\ell}(\lambda)$ can be expressed as $\mathcal{O}(\Delta_{\ell k}^{-2}) + \mathcal{O}(\delta^{\varepsilon-2})$, which holds uniformly in t, η, and λ.*

Proof. For part (a), it is easy to show that $|f_{t\ell}^{(r)}(\lambda)| \leq \sum_{j=0}^{\infty} j^r \eta^{j-1}$. By Lemma 1 in [27], $\sum_{j=0}^{\infty} t^r \eta^t = \mathcal{O}(\delta^{-r-1})$. Thus the expression for $f_{t\ell}^{(r)}(\lambda)$. The proof for $g_{t\ell}^{(r)}(\lambda)$ is analogous.

To prove part (b), consider the TSE

$$f_{t\ell}^{(r)}(\lambda) = f_{t\ell}^{(r)}(\omega_k) + (\lambda - \omega_k)\, f_{t\ell}^{(r+1)}(\lambda^*), \tag{6.6}$$

where λ^* lies between λ and ω_k. By Lemma 7(a), $f_{t\ell}^{(r+1)}(\lambda^*) = \mathcal{O}(\delta^{-r-2})$, so the second term in (6.6) is $\mathcal{O}(\delta^{q-r-2})$, hence the expression for $f_{t\ell}^{(r)}(\lambda)$. The proof for $g_{t\ell}^{(r)}(\lambda)$ is similar.

Finally, to prove part (c), consider the TSE

$$f_{t\ell}(\lambda) = f_{t\ell}(\lambda_k) + (\lambda - \lambda_k)\, \dot{f}_{t\ell}(\lambda^*),$$

where λ^* lies between λ and λ_k. Note that by Lemma 2, $\lambda^* \in \Lambda_{nk}$ and $\lambda - \lambda_k = \mathcal{O}(\delta^{\varepsilon})$, and by Lemma 7(a), $\dot{f}_{t\ell}(\lambda^*) = \mathcal{O}(\delta^{-2})$. Therefore,

$$f_{t\ell}(\lambda) = f_{t\ell}(\lambda_k) + \mathcal{O}(\delta^{\varepsilon-2}). \tag{6.7}$$

The assertion follows because the combination of Proposition 5(a), Lemma 6(a), and Lemma 7(b) implies that, for $\ell \neq k$, $f_{t\ell}(\lambda_k) = f_{t\ell}(\omega_k) + \mathcal{O}(1) = \mathcal{O}(\Delta_{\ell k}^{-2}) + \mathcal{O}(\Delta_{\ell k}^{-2}\eta^t) + \mathcal{O}(1) = \mathcal{O}(\Delta_{\ell k}^{-2})$. A similar argument can be applied to $g_{t\ell}(\lambda)$. Q.E.D.

Lemma 8 *Let the condition in Lemma 7 be satisfied. Assume that $n\eta^n = \mathcal{O}(1)$ and $\Delta_k^{-1}\delta \to 0$ as $n \to \infty$. Then, for $\ell' \neq k$, the following expressions are true.*

(a)

$$\sum_{t=1}^{n} c_{\ell'}(t)\, c_\ell(t)\, f_{t\ell}(\lambda_k) \;=\; \begin{cases} n\eta^{-1}\xi_{\ell k} + \mathcal{O}(\Delta_{\ell k}^{-4}\delta^2 n) + \mathcal{O}(\Delta_{\ell k}^{-2}\delta^{-1}) & \text{if } \ell' = \ell \neq k, \\ \min\{\mathcal{O}(\Delta_{\ell k}^{-2} n), \mathcal{O}(\Delta_{\ell k}^{-2}\Delta_{\ell'\ell}^{-1})\} + \mathcal{O}(\Delta_{\ell k}^{-2}\delta^{-1}) & \text{if } \ell' \neq \ell \neq k, \\ \mathcal{O}(\Delta_{\ell' k}^{-4}) + \mathcal{O}(\delta^{-1}) & \text{if } \ell' \neq \ell = k, \end{cases} \tag{6.8}$$

$$\sum_{t=1}^{n} c_{\ell'}(t)\, s_\ell(t)\, g_{t\ell}(\lambda_k) \;=\; \begin{cases} \mathcal{O}(\Delta_{\ell k}^{-2}\delta^{-1}) & \text{if } \ell' = \ell \neq k, \\ \text{same as in (6.8)} & \text{if } \ell' \neq \ell \neq k, \\ \mathcal{O}(\Delta_{\ell' k}^{-4}) + \mathcal{O}(\Delta_{\ell' k}^{-2}\delta^{-1}) & \text{if } \ell' \neq \ell = k, \end{cases} \tag{6.9}$$

where $\xi_{\ell k} := \frac{1}{8}\beta_\ell^2\{\cot(\frac{1}{2}\omega_{k\ell}^-) + \cot(\frac{1}{2}\omega_{k\ell}^+)\}$. In addition,

$$\sum_{t=1}^{n} c_k^2(t)\, f_{tk}(\lambda_k) = \mathcal{O}(\delta^2 n) + \mathcal{O}(\delta^{-1}). \tag{6.10}$$

(b)

$$\sum_{t=1}^{n} c_{\ell'}(t)\, c_\ell(t)\, \dot{f}_{t\ell}(\lambda_k) = \mathcal{O}(\delta^{-1}n) + \begin{cases} \mathcal{O}(\Delta_{\ell k}^{-4} n) + \mathcal{O}(\Delta_{\ell k}^{-2}\delta^{-2}) & \textit{if } \ell \neq k, \\ \mathcal{O}(\Delta_{\ell' k}^{-4}\delta^{-1}) + \mathcal{O}(\Delta_{\ell' k}^{-2}\delta^{-2}) & \textit{if } \ell = k, \end{cases} \tag{6.11}$$

$$\sum_{t=1}^{n} c_{\ell'}(t)\, s_\ell(t)\, \dot{g}_{t\ell}(\lambda_k) = \mathcal{O}(\delta^{-1}n) + \begin{cases} \mathcal{O}(\Delta_{\ell k}^{-4}\delta^{-1}) + \mathcal{O}(\Delta_{\ell k}^{-2}\delta^{-2}) & \textit{if } \ell' = \ell \neq k, \\ \textit{same as in } (6.11) & \textit{if } \ell' \neq \ell \neq k, \\ \mathcal{O}(\delta^{-2}) & \textit{if } \ell' \neq \ell = k. \end{cases} \tag{6.12}$$

(c)

$$\sum_{t=1}^{n} c_\ell(t)\, c_{\ell'}(t)\, f_{t\ell}(\lambda_k)\, \dot{f}_{t\ell'}(\lambda_k) = \mathcal{O}(\Delta_{\ell' k}^{-4} n) + \begin{cases} \mathcal{O}(\Delta_{\ell' k}^{-4}\Delta_{\ell k}^{-2} n) + \mathcal{O}(\Delta_{\ell' k}^{-2}\Delta_{\ell k}^{-2}\delta^{-2}) + \mathcal{O}(\Delta_{\ell k}^{-2}\delta^{-1} n) & \textit{if } \ell \neq k, \\ \mathcal{O}(\Delta_{\ell' k}^{-4}\delta^{-1}) + \mathcal{O}(\Delta_{\ell' k}^{-2}\delta^{-2}) + \mathcal{O}(\delta^{-1} n) & \textit{if } \ell = k, \end{cases} \tag{6.13}$$

$$\sum_{t=1}^{n} s_\ell(t)\, c_{\ell'}(t)\, g_{t\ell}(\lambda_k)\, \dot{f}_{t\ell'}(\lambda_k) = \mathcal{O}(\Delta_{\ell' k}^{-4} n) + \begin{cases} \mathcal{O}(\Delta_{\ell k}^{-6}\delta^{-1}) + \mathcal{O}(\Delta_{\ell k}^{-4}\delta^{-2}) + \mathcal{O}(\Delta_{\ell k}^{-2}\delta^{-1} n) & \textit{if } \ell' = \ell \neq k, \\ \textit{same as in } (6.13) & \textit{if } \ell' \neq \ell \neq k, \\ \mathcal{O}(\Delta_{\ell' k}^{-6}\delta^{-1}) + \mathcal{O}(\Delta_{\ell' k}^{-4}\delta^{-2}) + \mathcal{O}(\Delta_{\ell' k}^{-2}\delta^{-3}) + \mathcal{O}(\delta^{-2} n) & \textit{if } \ell' \neq \ell = k, \end{cases} \tag{6.14}$$

$$\sum_{t=1}^{n} c_\ell(t)\, s_{\ell'}(t)\, f_{t\ell}(\lambda_k)\, \dot{g}_{t\ell'}(\lambda_k) = \begin{cases} \textit{same as in } (6.14) & \textit{if } \ell' = \ell \neq k, \\ \textit{same as in } (6.13) & \textit{if } \ell' \neq \ell \neq k \textit{ or } \ell' \neq \ell = k, \end{cases} \tag{6.15}$$

$$\sum_{t=1}^{n} s_\ell(t)\, s_{\ell'}(t)\, g_{t\ell}(\lambda_k)\, \dot{g}_{t\ell'}(\lambda_k) = \begin{cases} \textit{same as in } (6.13) & \textit{if } \ell \neq k, \\ \textit{same as in } (6.14) & \textit{if } \ell = k. \end{cases} \tag{6.16}$$

Proof of (a). Define $\phi_{\ell'\ell}^{\pm}(t) := \omega_{\ell'\ell}^{\pm} t + \phi_{\ell'} \pm \phi_\ell$. To prove (6.8), it is helpful to note that

$$c_{\ell'}(t)\, c_\ell(t) = \tfrac{1}{2}\beta_{\ell'}\beta_\ell\{\cos(\phi_{\ell'\ell}^{+}(t)) + \cos(\phi_{\ell'\ell}^{-}(t))\}. \tag{6.17}$$

First, let $\ell' \neq \ell \neq k$. It can be shown, by Lemma 5(a), (6.5), and (6.17), that the product $c_{\ell'}(t)\, c_\ell(t)\, f_{t\ell}(\lambda_k)$, as a function of t, is a linear combination of $\cos(\phi^{\pm}_{\ell'\ell}(t))$, $\eta^t \exp(\pm it(\omega^{\pm}_{\ell'\ell} \pm \lambda^{+}_{k\ell}))$, $\Delta^{-2}_{\ell k} \cos(\phi^{\pm}_{\ell'\ell}(t))$, and $\Delta^{-2}_{\ell k}\eta^t \exp(\pm it(\omega^{\pm}_{\ell'\ell} \pm \lambda^{-}_{k\ell}))$. Note that $\sum \cos(\phi^{+}_{\ell'\ell}(t)) = \mathcal{O}(1)$. Note also that $\sum \cos(\phi^{-}_{\ell'\ell}(t)) = \min\{n, \mathcal{O}(\Delta^{-1}_{\ell'\ell})\}$ for $\ell' \neq \ell$ because $|\sum \exp(it\omega^{-}_{\ell'\ell})| \leq \min\{n, |\sin(\frac{1}{2}\omega^{-}_{\ell'\ell})|^{-1}\} = \min\{n, \mathcal{O}(\Delta^{-1}_{\ell'\ell})\}$. Combining these results with Lemma 3(a) yields (6.8) for $\ell' \neq \ell \neq k$.

Next, let $\ell' \neq \ell = k$. By (6.17) and Lemma 5(a), $c_{\ell'}(t)\, c_k(t)\, p_t(\lambda^{+}_{kk})$ is a linear combination of $\cos(\phi^{\pm}_{\ell' k}(t))$ and $\eta^t \exp(\pm it(\omega^{\pm}_{\ell' k} \pm \lambda^{+}_{kk}))$. This, combined (6.17) and the fact that $\sum \cos(\phi^{\pm}_{\ell' k}(t)) = \mathcal{O}(\Delta^{-1}_{\ell' k})$, leads to $\sum c_{\ell'}(t)\, c_k(t)\, p_t(\lambda^{+}_{kk}) = \mathcal{O}(\Delta^{-1}_{\ell' k}) + \mathcal{O}(\delta^{-1})$. Moreover, since $p_{0t}(0) = 0$, the TSE of $p_{0t}(\lambda - \omega_k)$ at $\lambda = \lambda_k$ takes the form

$$p_{0t}(\lambda^{-}_{kk}) = \dot{p}_{0t}(\zeta^*)\, \lambda^{-}_{kk}, \tag{6.18}$$

where ζ^* is between zero and λ^{-}_{kk}. By (6.17) and Lemma 4(d), $c_{\ell'}(t)\, c_k(t)\, \dot{p}_{0t}(\zeta^*)$ is a linear combination of $\cos(\phi^{\pm}_{\ell' k}(t))$ and $t^j \eta^t \exp(\pm it(\omega^{\pm}_{\ell' k} \pm \zeta^*))$ $(j = 0, 1)$. Note that summing the first expression over t yields $\mathcal{O}(\Delta^{-1}_{\ell' k})$ and summing the second expression, by Lemma 4 in [27], leads to $\mathcal{O}(g^{j+1}(\omega^{\pm}_{\ell' k} \pm \zeta^*))$ under the assumption $n\eta^n = \mathcal{O}(1)$. Since $\zeta^* = \mathcal{O}(\delta^2)$, one obtains $(\omega^{-}_{\ell' k} \pm \zeta^*)\delta^{-1} \to \infty$ and $\omega^{-}_{\ell' k} \pm \zeta^* = \mathcal{O}(\Delta_{\ell' k})$. Moreover, $\omega^{+}_{\ell' k} \pm \zeta^*$ can be bounded away from zero and $\pm 2\pi$. By Lemma 4, $g(\omega^{-}_{\ell' k} \pm \zeta^*) = \mathcal{O}(\Delta^{-2}_{\ell' k})$ and $g(\omega^{+}_{\ell' k} \pm \zeta^*) = \mathcal{O}(1)$. Therefore, $\sum c_{\ell'}(t)\, c_k(t)\, \dot{p}_{0t}(\zeta^*) = \mathcal{O}(\Delta^{-4}_{\ell' k})$. Substituting this result in (6.18) yields $\sum c_{\ell'}(t)\, c_k(t)\, p_{0t}(\lambda^{-}_{kk}) = \mathcal{O}(\Delta^{-4}_{\ell' k}\delta^2)$. Furthermore, since $g(\lambda^{-}_{kk}) = \mathcal{O}(\delta^{-2})$, one obtains $\sum c_{\ell'}(t)\, c_k(t)\, p_t(\lambda^{-}_{kk}) = \mathcal{O}(\Delta^{-4}_{\ell' k})$. This, combined with (6.5), proves (6.8) for $\ell' \neq \ell = k$.

Now consider the remaining case of $\ell' = \ell \neq k$. Note that $c^2_\ell(t) = \frac{1}{2}\beta^2_\ell\{1 + \cos(\phi^{+}_{\ell\ell}(t))\}$. As in the first case, one obtains $\sum \cos(\phi^{+}_{\ell\ell}(t))\, f_{t\ell}(\lambda_k) = \mathcal{O}(\Delta^{-2}_{\ell k}\delta^{-1})$. Therefore, it suffices to show that

$$\sum_{t=1}^{n} f_{t\ell}(\lambda_k) = n\eta^{-1}\xi_{\ell k} + \mathcal{O}(\Delta^{-4}_{\ell k}\delta^2 n) + \mathcal{O}(\Delta^{-2}_{\ell k} n^{-1}). \tag{6.19}$$

To this end, one first obtains, by a straightforward calculation based on (6.1), that

$$\sum_{t=1}^{n} p_t(\lambda) = g(\lambda)\, \{n \sin\lambda - (1 - \eta^2)\, p_n(\lambda) - \eta^{n-1}\sin(n\lambda)\}. \tag{6.20}$$

Since $n\eta^n = \mathcal{O}(1)$ and $g(\lambda^{+}_{k\ell}) = \mathcal{O}(1)$ by Lemma 4(a), $\sum p_t(\lambda^{+}_{k\ell}) = n g(\lambda^{+}_{k\ell}) \sin(\lambda^{+}_{k\ell}) + \mathcal{O}(\delta) + \mathcal{O}(n^{-1})$. Again, by Lemma 4(a), $g(\lambda^{+}_{k\ell}) = \frac{1}{4}\eta^{-1}(\sin(\frac{1}{2}\lambda^{+}_{k\ell}))^{-2} + \mathcal{O}(\delta^2)$. Therefore, $\sum p_t(\lambda^{+}_{k\ell}) = \frac{1}{2}\eta^{-1} n \cot(\frac{1}{2}\lambda^{+}_{k\ell}) + \mathcal{O}(\delta^2 n) + \mathcal{O}(\delta) + \mathcal{O}(n^{-1})$. Moreover, since $\lambda^{+}_{k\ell} = \omega^{+}_{k\ell} + \mathcal{O}(\delta^2)$ and $\lambda^{+}_{k\ell}$ is bounded away from zero and $\pm 2\pi$, one obtains $\cot(\frac{1}{2}\lambda^{+}_{k\ell}) = \cot(\frac{1}{2}\omega^{+}_{k\ell}) + \mathcal{O}(\delta^2)$. Therefore,

$$\sum_{t=1}^{n} p_t(\lambda^{+}_{k\ell}) = \tfrac{1}{2}\eta^{-1} n \cot(\tfrac{1}{2}\omega^{+}_{k\ell}) + \mathcal{O}(\delta^2 n) + \mathcal{O}(\delta) + \mathcal{O}(n^{-1}). \tag{6.21}$$

On the other hand, by Lemma 4(b), $g(\lambda_{k\ell}^-) = \mathcal{O}(\Delta_{\ell k}^{-2})$, and by Lemma 4 in [27], $p_n(\lambda_{k\ell}^-) = \mathcal{O}(g(\lambda_{k\ell}^-))$. Therefore, it follows from (6.20) that $\sum p_t(\lambda_{k\ell}^-) = ng(\lambda_{k\ell}^-)\sin(\lambda_{k\ell}^-) + \mathcal{O}(\Delta_{\ell k}^{-4}\delta) + \mathcal{O}(\Delta_{\ell k}^{-2}n^{-1})$. By Lemma 4(b), $g(\lambda_{k\ell}^-) = \frac{1}{4}\eta^{-1}(\sin(\frac{1}{2}\lambda_{k\ell}^-))^{-2} + \mathcal{O}(\Delta_{\ell k}^{-4}\delta^2)$. Moreover, since $\lambda_{k\ell}^- - \omega_{k\ell}^- = \mathcal{O}(\delta^2)$, it can be shown, by TSE, that $\cot(\frac{1}{2}\lambda_{k\ell}^-) = \cot(\frac{1}{2}\omega_{k\ell}^-) + \mathcal{O}(\Delta_{\ell k}^{-2}\delta^2)$. Therefore,

$$\sum_{t=1}^{n} p_t(\lambda_{k\ell}^-) = \tfrac{1}{2}\eta^{-1}n\cot(\tfrac{1}{2}\omega_{k\ell}^-) + \mathcal{O}(\Delta_{\ell k}^{-4}\delta^2 n) + \mathcal{O}(\Delta_{\ell k}^{-2}n^{-1}). \tag{6.22}$$

Combining (6.21) and (6.22) with (6.5) leads to (6.19) and thus proves (6.8) for $\ell' = \ell \neq k$.

To prove (6.9), it is helpful to observe that

$$c_{\ell'}(t)\,s_\ell(t) = \tfrac{1}{2}\beta_{\ell'}\beta_\ell\{\sin(\phi_{\ell'\ell}^+(t)) - \sin(\phi_{\ell'\ell}^-(t))\}. \tag{6.23}$$

For $\ell \neq k$, it can be shown from (6.23) and Lemma 4(d) that $c_{\ell'}(t)\,s_\ell(t)\,q_{0t}(\lambda_{k\ell}^\pm)$ is a linear combination of $\sin(\phi_{\ell'\ell}^\pm(t))$ and $\eta^t\exp(\pm it(\omega_{\ell'\ell}^\pm \pm \lambda_{k\ell}^\pm))$. Note that $\sum\sin(\phi_{\ell'\ell}^-(t)) = \min\{n, \mathcal{O}(\Delta_{\ell'\ell}^{-1})\}$ if $\ell' \neq \ell$ and $\sum\sin(\phi_{\ell'\ell}^-(t)) = 0$ if $\ell' = \ell$. Note also that $\sum\sin(\phi_{\ell'\ell}^+(t)) = \mathcal{O}(1)$ and $\sum\eta^t\exp(\pm it\lambda) = \mathcal{O}(\delta^{-1})$ for any ℓ' and λ. Therefore, $\sum c_{\ell'}(t)\,s_\ell(t)\,q_{0t}(\lambda_{k\ell}^\pm)$ takes the form $\min\{n, \mathcal{O}(\Delta_{\ell'\ell}^{-1})\} + \mathcal{O}(\delta^{-1})$ if $\ell' \neq \ell$ and the form $\mathcal{O}(\delta^{-1})$ if $\ell' = \ell$. By Lemma 4(a) and (b), $g(\lambda_{k\ell}^+) = \mathcal{O}(1)$ and $g(\lambda_{k\ell}^-) = \mathcal{O}(\Delta_{\ell k}^{-2})$. Combining these results with (6.5) and (6.1) proves (6.9) for $\ell \neq k$.

Now consider $\ell' \neq \ell = k$. Since $g(\lambda_{kk}^+) = \mathcal{O}(1)$ and since $c_{\ell'}(t)\,s_k(t)\,q_{0t}(\lambda_{kk}^+)$ is a linear combination of $\sin(\phi_{\ell' k}^\pm(t))$ and $\eta^t\exp(\pm it(\omega_{\ell' k}^\pm \pm \lambda_{kk}^+))$, it follows that $\sum c_{\ell'}(t)\,s_k(t)\,q_t(\lambda_{kk}^+) = \mathcal{O}(\Delta_{\ell' k}^{-1}) + \mathcal{O}(\delta^{-1})$. Moreover, the TSE of $q_{0t}(\lambda_{kk}^-)$ can be expressed as

$$q_{0t}(\lambda_{kk}^-) = q_{0t}(0) + \dot{q}_{0t}(\zeta^*)\,\lambda_{kk}^-,$$

where ζ^* lies between zero and λ_{kk}^-. By (6.23) and Lemma 4(d), $c_{\ell'}(t)\,s_k(t)\,\dot{q}_{0t}(\zeta^*)$ is a linear combination of $\sin(\phi_{\ell' k}^\pm(t))$ and $t^j\eta^t\exp(\pm it(\omega_{\ell' k}^\pm \pm \zeta^*))$ $(j = 0, 1)$. As with the sum of $c_{\ell'}(t)\,c_\ell(t)\,\dot{p}_{0t}(\zeta^*)$, one can show that $\sum c_{\ell'}(t)\,s_k(t)\,\dot{q}_{0t}(\zeta^*) = \mathcal{O}(\Delta_{\ell' k}^{-4})$. Moreover, since $q_{0t}(0) = \delta\eta^{-1}(1-\eta^t)$, the product $c_{\ell'}(t)\,s_k(t)\,q_{0t}(0)$ is a linear combination of $\delta\sin(\phi_{\ell' k}^\pm(t))$ and $\delta\eta^t\sin(\phi_{\ell' k}^\pm(t))$. So, by Lemma 3(b) and Lemma 4(c), $\sum c_{\ell'}(t)\,s_k(t)\,q_{0t}(0) = \mathcal{O}(\Delta_{\ell' k}^{-2}\delta)$. Combining these with $\lambda_{kk}^- = \mathcal{O}(\delta^2)$ and $g(\lambda_{kk}^-) = \mathcal{O}(\delta^{-2})$ yields $\sum c_{\ell'}(t)\,s_k(t)\,q_t(\lambda_{kk}^-) = \mathcal{O}(\Delta_{\ell' k}^{-2}\delta^{-1}) + \mathcal{O}(\Delta_{\ell' k}^{-4})$, and thus (6.9) for $\ell = k$.

To prove (6.10), it is important to note that since $\lambda_{kk}^- = \mathcal{O}(\delta^2)$ and $n\eta^n = \mathcal{O}(1)$, one can write $g(\lambda_{kk}^-) = \mathcal{O}(\delta^{-2})$, $\sin(\lambda_{kk}^-) = \mathcal{O}(\delta^2)$, $\eta^{n-1}\sin(n\lambda_{kk}^-) = \mathcal{O}(\delta^2)$, and $\eta^n\sin((n-1)\lambda_{kk}^-) = \mathcal{O}(\delta^2)$. This, combined with (6.1) and (6.3), implies that $p_n(\lambda_{kk}^-) = \mathcal{O}(1)$. Substituting these results in (6.20) leads to $\sum p_t(\lambda_{kk}^-) = ng(\lambda_{kk}^-)\sin(\lambda_{kk}^-) + \mathcal{O}(\delta^{-1})$. Furthermore, since Proposition 5(a) implies that $\lambda_{kk}^- = -\frac{1}{2}\eta^{-1}\delta^2\cot\omega_k + \mathcal{O}(\delta^4)$, one obtains $\sin(\lambda_{kk}^-) = \lambda_{kk}^- + \mathcal{O}((\lambda_{kk}^-)^3) = -\frac{1}{2}\eta^{-1}\delta^2\cot\omega_k + \mathcal{O}(\delta^4)$. In addition, one can write $g(\lambda_{kk}^-) =$

$\delta^{-2}+\mathcal{O}(1)$. Combining these results yields

$$\sum_{t=1}^{n} p_t(\lambda_{kk}^-) = -\tfrac{1}{2}\eta^{-1} n \cot\omega_k + \mathcal{O}(\delta^2 n) + \mathcal{O}(\delta^{-1}).$$

This, coupled with (6.21), which remains valid for $\ell = k$, gives rise to

$$\sum_{t=1}^{n} f_{tk}(\lambda_k) = \mathcal{O}(\delta^2 n) + \mathcal{O}(\delta^{-1}).$$

Further, by using (6.18) and Lemma 4(d), one can show that $\sum \cos(\phi_{kk}^+(t))\, p_t(\lambda_{kk}^-) = \mathcal{O}(1)$; by Lemma 3(a) and Lemma 5(a), one obtains $\sum \cos(\phi_{kk}^+(t))\, p_t(\lambda_{kk}^+) = \mathcal{O}(\delta^{-1})$. Therefore,

$$\sum_{t=1}^{n} \cos(\phi_{kk}^+(t))\, f_{tk}(\lambda_k) = \mathcal{O}(\delta^{-1}).$$

Equation (6.10) follows immediately upon noting that $c_k^2(t) = \frac{1}{2}\{1+\cos(\phi_{kk}^+(t))\}$. Q.E.D.

Proof of (b). To show (6.11), it suffices to note that by Lemma 7(b),

$$\dot f_{t\ell}(\lambda_k) = \dot f_{t\ell}(\omega_k) + \mathcal{O}(\delta^{-1}). \tag{6.24}$$

For $\ell \neq k$, one can show, by using Lemma 6(a) and (6.17), that $c_{\ell'}(t)\, c_\ell(t)\, \dot f_{t\ell}(\omega_k)$ is a linear combination of $\cos(\phi_{\ell'\ell}^\pm(t))$, $t^j \eta^t \exp(\pm it(\omega_{\ell'\ell}^\pm \pm \omega_{k\ell}^+))$, $\Delta_{\ell k}^{-2(2-j)} \cos(\phi_{\ell'\ell}^\pm(t))$, and $\Delta_{\ell k}^{-2(2-j)} t^j \eta^t \exp(\pm it(\omega_{\ell'\ell}^\pm \pm \omega_{k\ell}^-))$ $(j=0,1)$. This, combined with (6.24), Lemma 3(a), and the fact that $\sum \cos(\phi_{\ell'\ell}^\pm(t)) = \mathcal{O}(n)$, leads to (6.11) for $\ell' \neq \ell \neq k$.

For $\ell' = \ell \neq k$, it follows from Lemma 6(a) that $\sum \cos(\phi_{\ell\ell}^+(t))\, \dot f_{t\ell}(\omega_k) = \mathcal{O}(\Delta_{\ell k}^{-4}\delta^{-1}) + \mathcal{O}(\Delta_{\ell k}^{-2}\delta^{-2})$ due to the fact that $\sum \cos(\phi_{\ell\ell}^+(t)) = \mathcal{O}(1)$. It also follows from Lemma 6(a) that $\sum \dot f_{t\ell}(\omega_k) = \mathcal{O}(\Delta_{\ell k}^{-4} n) + \mathcal{O}(\Delta_{\ell k}^{-4}\delta^{-1}) + \mathcal{O}(\Delta_{\ell k}^{-2}\delta^{-2}) = \mathcal{O}(\Delta_{\ell k}^{-4} n) + \mathcal{O}(\Delta_{\ell k}^{-2}\delta^{-2})$. Combining these results with (6.17) and (6.24) leads to (6.11) for $\ell' = \ell \neq k$.

In the remaining case of $\ell' \neq \ell = k$, Lemma 6(b) ensures that $c_{\ell'}(t)\, c_k(t)\, \dot f_{tk}(\omega_k)$ is a linear combination of $\cos(\phi_{\ell' k}^\pm(t))$, $\delta^{-2}\cos(\phi_{\ell' k}^\pm(t))$, $\delta^{-2+j} t^j \eta^t \cos(\phi_{\ell' k}^\pm(t))$, and $t^j \eta^t \exp(\pm it(\omega_{\ell' k}^\pm \pm \omega_{kk}^+))$ $(j=0,1)$. The assertion follows immediately from the fact that $\sum t^j \eta^t \cos(\phi_{\ell' k}^-(t)) = \mathcal{O}(\Delta_{\ell' k}^{-2(j+1)})$ by Lemma 4(c), $\sum t^j \eta^t \exp(\pm it(\omega_{\ell' k}^\pm \pm \omega_{kk}^+)) = \mathcal{O}(\delta^{-j-1})$ by Lemma 3(a), and $\sum \cos(\phi_{\ell' k}^+(t)) = \mathcal{O}(1)$.

Equation (6.12) can be proved similarly. First, by Lemma 7(b),

$$\dot g_{t\ell}(\lambda_k) = \dot g_{t\ell}(\omega_k) + \mathcal{O}(\delta^{-1}). \tag{6.25}$$

It follows from Lemma 6(a) and (6.23) that $c_{\ell'}(t)\, s_\ell(t)\, \dot g_{t\ell}(\omega_k)$ is a linear combination of $\sin(\phi_{\ell'\ell}^\pm(t))$, $t^j \eta^t \times \exp(\pm it(\omega_{\ell'\ell}^\pm \pm \omega_{k\ell}^+))$, $\Delta_{\ell k}^{-2(2-j)} \sin(\phi_{\ell'\ell}^\pm(t))$, and $\Delta_{\ell k}^{-2(2-j)} t^j \eta^t \exp(\pm it(\omega_{\ell'\ell}^\pm \pm \omega_{k\ell}^-))$ $(j=0,1)$. Therefore, for

$\ell' \neq \ell \neq k$, $\sum c_{\ell'}(t)\,s_\ell(t)\,\dot{g}_{t\ell}(\omega_k)$ has the same expression as $\sum c_{\ell'}(t)\,c_\ell(t)\,\dot{f}_{t\ell}(\omega_k)$. For $\ell' = \ell \neq k$, one can show by using Lemma 6(a) that $\sum \sin(\phi^+_{\ell\ell}(t))\,\dot{g}_{t\ell}(\omega_k) = \mathcal{O}(\Delta^{-4}_{\ell k}\delta^{-1}) + \mathcal{O}(\Delta^{-2}_{\ell k}\delta^{-2})$. This, combined with (6.23) and the fact that $\sin(\phi^-_{\ell\ell}(t)) = 0$ leads to the assertion. Finally, for $\ell' \neq \ell = k$, it follows from Lemma 6(b) and (6.23) that $c_{\ell'}(t)\,s_k(t)\,\dot{g}_{tk}(\omega_k)$ is a linear combination of $\sin(\phi^\pm_{\ell' k}(t))$ and $t^j\eta^t\exp(\pm it(\omega^\pm_{\ell' k} \pm \omega^+_{kk}))$ $(j = 0,1)$. Since summing the first expression over t yields $\mathcal{O}(\Delta^{-1}_{\ell' k})$ and summing the second one leads to $\mathcal{O}(\delta^{-2})$ by Lemma 3(a), the proof is complete upon noting that $\mathcal{O}(\Delta^{-1}_{\ell' k}) + \mathcal{O}(\delta^{-2}) = \mathcal{O}(\delta^{-2})$. Q.E.D.

Proof of (c). Since $\lambda_k - \omega_k = \mathcal{O}(\delta^2)$, one obtains, by Lemma 7(b),

$$f_{t\ell}(\lambda_k) = f_{t\ell}(\omega_k) + \mathcal{O}(1). \tag{6.26}$$

For $\ell \neq k$, Lemma 6(a) ensures that $f_{t\ell}(\omega_k)\,\dot{f}_{t\ell'}(\omega_k)$ is a linear combination of functions that take the form $\Delta^{-2p(2-j)}_{\ell' k}\Delta^{-2q}_{\ell k}$, $\Delta^{-2p(2-j)}_{\ell' k}\Delta^{-2q}_{\ell k}\eta^t\exp(\pm it\lambda)$, and $\Delta^{-2p(2-j)}_{\ell' k}\Delta^{-2q}_{\ell k}t^j\eta^{(r+1)t}\exp(\pm it\lambda)$ $(p,q,r,j = 0,1)$. Since $\sum\cos(\phi^\pm_{\ell'\ell}(t)) = \mathcal{O}(n)$, it follows from (6.17) and Lemma 3(a) that $\sum c_\ell(t)\,c_{\ell'}(t)\,f_{t\ell}(\omega_k)\,\dot{f}_{t\ell'}(\omega_k) = \mathcal{O}(\Delta^{-4}_{\ell' k}\Delta^{-2}_{\ell k}n) + \mathcal{O}(\Delta^{-4}_{\ell' k}\Delta^{-2}_{\ell k}\delta^{-1}) + \mathcal{O}(\Delta^{-2}_{\ell' k}\Delta^{-2}_{\ell k}\delta^{-2}) = \mathcal{O}(\Delta^{-4}_{\ell' k}\Delta^{-2}_{\ell k}n) + \mathcal{O}(\Delta^{-2}_{\ell' k}\Delta^{-2}_{\ell k}\delta^{-2})$. For $\ell' \neq \ell = k$, Lemma 6 implies that $f_{tk}(\omega_k)\,\dot{f}_{t\ell'}(\omega_k)$ is a linear combination of functions that take the form $\Delta^{-2p(2-j)}_{\ell' k}$, $\Delta^{-2p(2-j)}_{\ell' k}\eta^t\exp(\pm it\lambda)$, and $\Delta^{-2p(2-j)}_{\ell' k}t^j\eta^{(r+1)t}\exp(\pm it\lambda)$ $(p,r,j = 0,1)$. This result, combined with (6.17) and the fact that $\sum\cos(\phi^\pm_{\ell' k}(t)) = \mathcal{O}(\Delta^{-1}_{\ell' k})$, yields $\sum c_k(t)\,c_{\ell'}(t)\,f_{tk}(\omega_k)\,\dot{f}_{t\ell'}(\omega_k) = \mathcal{O}(\Delta^{-5}_{\ell' k}) + \mathcal{O}(\Delta^{-4}_{\ell' k}\delta^{-1}) + \mathcal{O}(\Delta^{-2}_{\ell' k}\delta^{-2}) = \mathcal{O}(\Delta^{-4}_{\ell' k}\delta^{-1}) + \mathcal{O}(\Delta^{-2}_{\ell' k}\delta^{-2})$. Further, by Lemma 6, $\dot{f}_{t\ell'}(\omega_k) = \mathcal{O}(\Delta^{-4}_{\ell' k}) + \sum_{j=0}^{1}\mathcal{O}(\Delta^{-2(2-j)}_{\ell' k}t^j\eta^t)$, $f_{t\ell}(\omega_k) = \mathcal{O}(\Delta^{-2}_{\ell k})$ if $\ell \neq k$, and $f_{tk}(\omega_k) = \mathcal{O}(1)$. Combining these results with (6.24) and (6.26) leads to (6.13).

Similarly, to prove (6.14), one first observes that

$$g_{t\ell}(\lambda_k) = g_{t\ell}(\omega_k) + \mathcal{O}(1). \tag{6.27}$$

For $\ell' \neq \ell \neq k$, one can show, by using Lemma 6(a), (6.23), and the fact that $\sum\sin(\phi^\pm_{\ell'\ell}(t)) = \mathcal{O}(n)$, that $\sum s_\ell(t)\,c_{\ell'}(t)\,g_{t\ell}(\omega_k)\,\dot{f}_{t\ell'}(\omega_k)$ has the same asymptotic expression as $\sum c_\ell(t)\,c_{\ell'}(t)\,f_{t\ell}(\omega_k)\,\dot{f}_{t\ell'}(\omega_k)$. In the case of $\ell' = \ell \neq k$, since $\sum\sin(\phi^\pm_{\ell\ell}(t)) = \mathcal{O}(1)$, it follows from Lemma 6(a) and (6.23) that $\sum s_\ell(t)\,c_\ell(t)\,g_{t\ell}(\omega_k)\times \dot{f}_{t\ell}(\omega_k) = \mathcal{O}(\Delta^{-6}_{\ell k}\delta^{-1}) + \mathcal{O}(\Delta^{-4}_{\ell k}\delta^{-2})$. Finally, for $\ell' \neq \ell = k$, Lemma 6 guarantees that $g_{tk}(\omega_k)\,\dot{f}_{t\ell'}(\omega_k)$ is a linear combination of functions that take the form $\Delta^{-2p(2-j)}_{\ell' k}\delta^{-q}$, $\Delta^{-2p(2-j)}_{\ell' k}\delta^{-1}\eta^t$, $\Delta^{-2p(2-j)}_{\ell' k}\eta^t\exp(\pm it\lambda)$, and $\Delta^{-2p(2-j)}_{\ell' k}\delta^{-q}t^j\eta^{(r+1)t}\exp(\pm it\lambda)$ $(p,q,r,j = 0,1)$. By Lemma 4(c), one can write $\sum\eta^t\sin(\phi^\pm_{\ell' k}(t)) = \mathcal{O}(\Delta^{-2}_{\ell' k})$. Combining these results with (6.23), Lemma 3(a), and the fact that $\sum\sin(\phi^\pm_{\ell' k}(t)) = \mathcal{O}(\Delta^{-1}_{\ell' k})$ leads to $\sum s_k(t)\,c_{\ell'}(t)\,g_{tk}(\omega_k)\,\dot{f}_{t\ell'}(\omega_k) = \mathcal{O}(\Delta^{-6}_{\ell' k}\delta^{-1}) + \mathcal{O}(\Delta^{-4}_{\ell' k}\delta^{-2}) + \mathcal{O}(\Delta^{-2}_{\ell' k}\delta^{-3})$. Moreover, by Lemma 6, $g_{t\ell}(\omega_k) = \mathcal{O}(\Delta^{-1}_{\ell k})$ if $\ell \neq k$, and $g_{tk}(\omega_k) = \mathcal{O}(\delta^{-1})$. Combining these results with (6.24) and (6.27) proves (6.14).

Note that $f_{t\ell}(\omega_k)\,\dot{g}_{t\ell'}(\omega_k)$ is a linear combination of the functions that constitute $g_{t\ell}(\omega_k)\,\dot{f}_{t\ell'}(\omega_k)$ for $\ell' = \ell \neq k$ as well as the functions that constitute $f_{t\ell}(\omega_k)\,\dot{f}_{t\ell'}(\omega_k)$ for any ℓ and any $\ell' \neq k$. Moreover, $\sum \sin(\phi^{\pm}_{\ell'\ell}(t)) = \mathcal{O}(n)$ for $\ell' \neq \ell$, and $\sum \sin(\phi^{\pm}_{\ell\ell}(t)) = \mathcal{O}(1)$. Combining these results with (6.23) and Lemma 6 proves (6.15).

Finally, one can prove (6.16) for $\ell' \neq \ell \neq k$ by observing that $\sum s_\ell(t)\,s_{\ell'}(t)\,g_{t\ell}(\omega_k)\,\dot{g}_{t\ell'}(\omega_k)$ has the same expression as $\sum s_\ell(t)\,c_{\ell'}(t)\,g_{t\ell}(\omega_k)\,\dot{f}_{t\ell'}(\omega_k)$. The assertion for $\ell' = \ell \neq k$ follows from the fact that $g_{t\ell}(\omega_k)\,\dot{g}_{t\ell}(\omega_k)$ is a linear combination of the same functions which form $f_{t\ell}(\omega_k)\,\dot{f}_{t\ell}(\omega_k)$ and the fact that $\sum \cos(\phi^{\pm}_{\ell\ell}(t)) = \mathcal{O}(n)$. The case of $\ell' \neq \ell = k$ can be proved by using the fact that $g_{tk}(\omega_k)\,\dot{g}_{t\ell'}(\omega_k)$ is a linear combination of the same functions which constitute $g_{tk}(\omega_k)\,\dot{f}_{t\ell'}(\omega_k)$ and the fact that $\sum \cos(\phi^{\pm}_{\ell' k}(t)) = \mathcal{O}(\Delta^{-1}_{\ell' k})$. Q.E.D.

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